Potts model with invisible states: critical behaviour on a scale-free network

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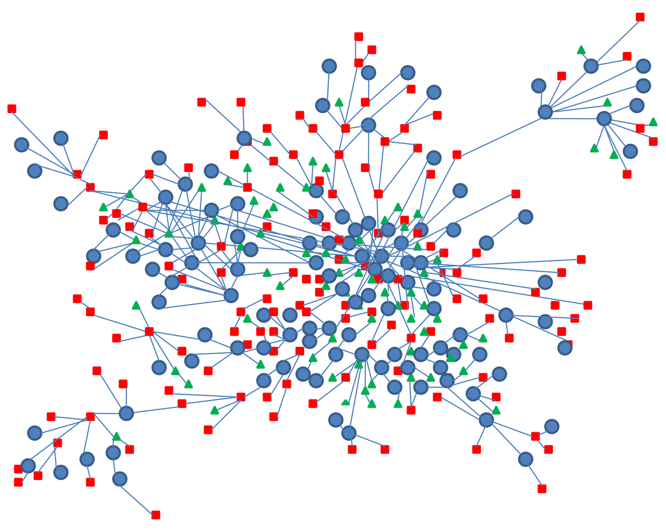
Different models are proposed to understand the phase transitions in (magnetic) systems through the prism of competition between the energy and the entropy. One of such models is a q-state Potts model with invisible states. This model introduces r invisible states such that if spin lies in one of them, it does not interact with the rest of the system [1]. We consider such a model on an annealed scale-free network where the probability of a randomly chosen vertex having a degree k is governed by the power-law $P(k) \propto k^{-\lambda}$. We confirm the previously obtained results and conclusions, namely that the number of invisible states can change the universality class of models on graphs. In particular, on a complete graph [2] or even on a scale-free network [3], when the degree distribution decay exponent plays an important role. Here, after numerical analysis of the free energy of the Potts model with invisible states on a scale-free network, we conclude that q, r, λ play a role of global parameters that influence the critical behaviour of the system [4]. The phase diagram in q, r, λ space can be separated into two main regions. In the region $1 < q \le 2, 3 \le \lambda$ and $q > 2$, $3 \le \lambda \le \lambda_c(q)$ we found two marginal values of r dependent on λ and q. They divide the phase diagram into three domains with different critical behaviour. For $q > 2, \lambda > \lambda_c(q)$ only the first order phase transition is observed regardless of values of r.

MODEL



Scale-free networks: the probability of a randomly chosen vertex having a degree k decays as a power-law

$$P(k) \propto k^{-\lambda}. \quad (1)$$

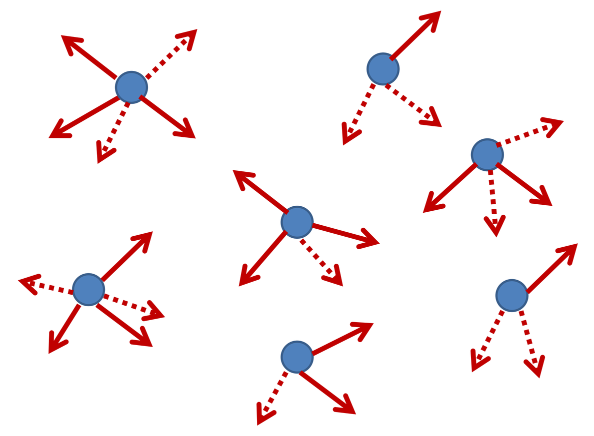
Hamiltonian of the q-state Potts model with invisible states:

$$-H(q,r) = \sum_{<i,j>} J_{ij} \sum_{\alpha=1}^{q} \delta_{S_i,\alpha}\delta_{\alpha,S_j} + h\sum_{i=1}^{N}\delta_{S_i,1}, \quad (2)$$

where $S_i = (1, ..., q, q+1, ..., q+r)$ is the value of the Potts variable on node i. We define q standard (visible) states and r invisible states (spins in such states don't interact with each other). The external magnetic field h is introduced to favour the first visible state.

Partition function:

$$Z = \prod_i \sum_{S_i=1}^{q+r} \exp(-\beta H). \quad (3)$$



Spins $S_i = (1, 2, ..., q, q+1, ...q+r)$ with standard (solid lines) and invisible (dashed lines) states. There is an interaction only between spins in the same standard states. The invisible states do not contribute to the interacting energy, but contribute to entropy)

METHOD

We follow the general steps of MFA, namely we use a variant of a mean-field approach with local order parameters m_{1i}, m_{2i}. The network as a whole is described by the global order parameters m_1, m_2. We introduce them as a linear combination of the local order parameters with weights proportional to the degree of a node k:

$$m_1 = \frac{\sum_i k_i m_{1i}}{\sum_i k_i}, \qquad m_2 = \frac{\sum_i k_i m_{2i}}{\sum_i k_i}. \quad (4)$$

Considering the free energy per site in the thermodynamic limit ($N \to \infty$) finally for the free energy per site we obtain:

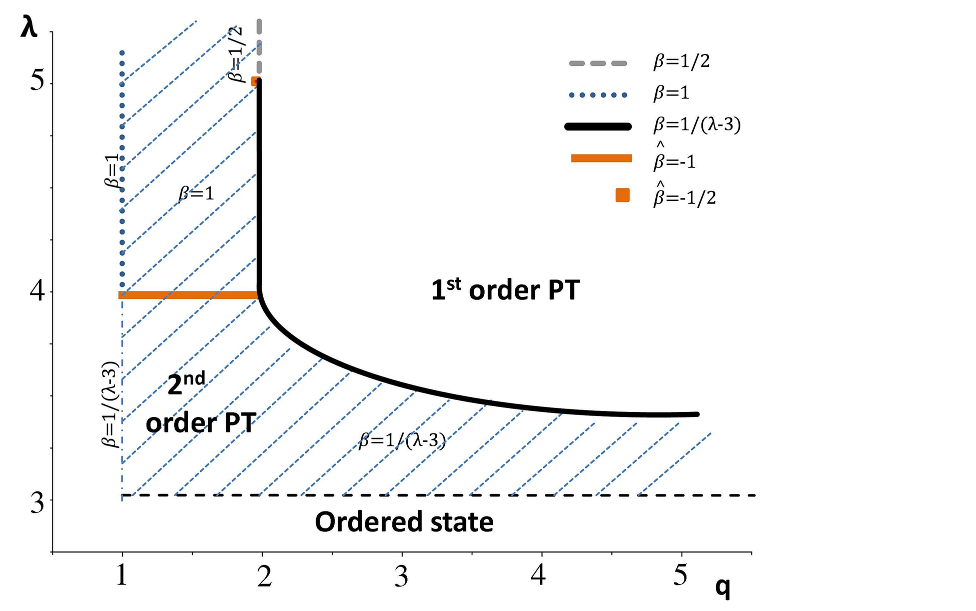
$$f(m_1, m_2) = \frac{J\langle k\rangle}{(q+r)^2}\Big((rm_2 + 1 + (q-1)m_1)^2 + (q-1)(rm_2 + 1 - (r+1)m_1)^2\Big) - \frac{1}{\beta}\int_2^\infty dkP(k)\ln\Big(e^{\beta(h+\frac{kJ}{q+r}(m_1(q-1)+1+rm_2))} + (q-1)e^{\frac{\beta Jk}{q+r}(m_2 r+1-(r+1)m_1)} + r\Big), \quad (5)$$

where $P(k)$ is the node degree distribution (1). We adopt two minimization methods – the simplex method [5] and the simulated annealing [6] to get the numerical results. We fix the values of q, r and λ and sweep through a certain region of temperatures to calculate the values of m_1 and m_2 to get the free energy minimum and investigate it in more details.

KNOWN RESULTS

Potts model on a scale free network

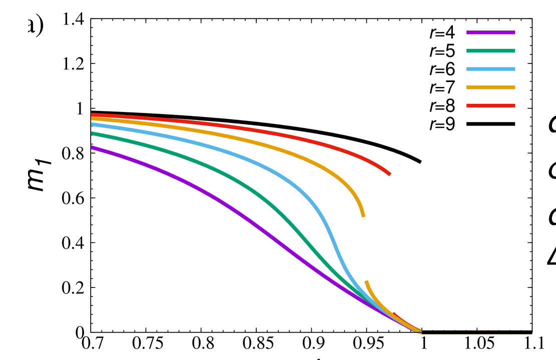
(M. Krasnytska, B. Berche, Yu. Holovatch. Condens. Matter Phys., vol. 16, No. 2, 23602 (2013): arXiv:1302.3386)



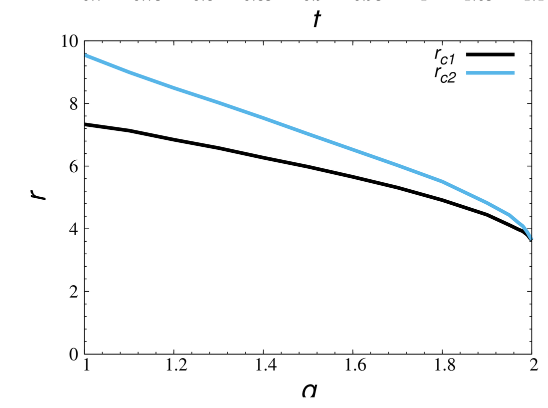
The phase diagram of the Potts model without invisible states on a scale-free network. β is a critical exponent for the order parameter: $m \sim \tau^{\beta}$.

Potts model with invisible states on a complete graph

(M. Krasnytska, P. Sarkanych, B. Berche, Yu. Holovatch, R. Kenna. J. Phys. A: Math. Theor. vol. 49, 255001 (2016): arXiv:1512.03635)



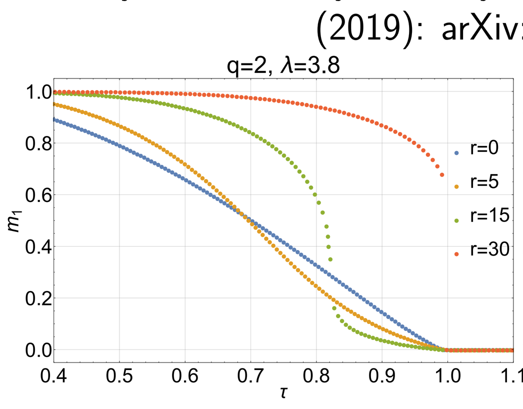
$q = 1.2$: Dependencies of the order parameter m_1 on the reduced temperature t for $r = 4, 5, 6, 7, 8, 9$



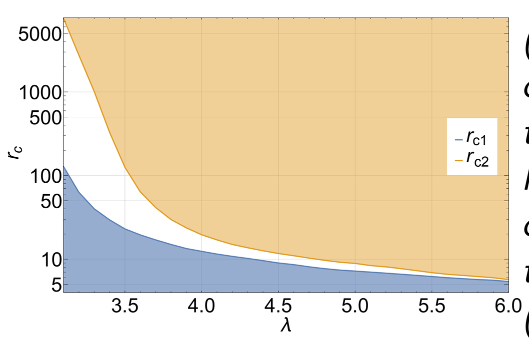
Two **marginal dimensions**, *$r = r_{c1}$ and $r = r_{c2}$. Transition is of the 1st order in the region above the blue curve and of the 2nd order below the black curve. Both 1st and2nd order phase transitions occur in the intermediate region*

Ising model with invisible states on scale-free networks

(P. Sarkanych, M. Krasnytska. Phys. Lett. A, vol. 383, Issue 27, 125844 (2019): arXiv:1904.06563)



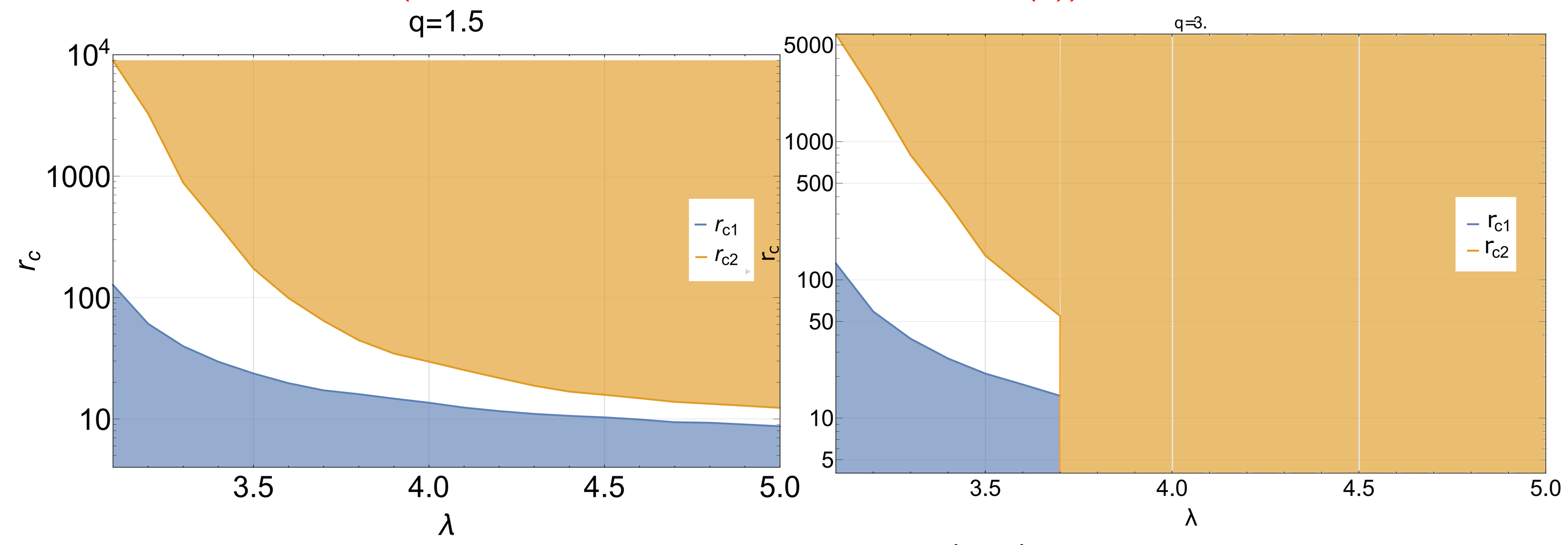
$q = 2$, $\lambda = 3.8$: Order parameter m_1 as a function of the reduced temperature $\tau = T/T_c$ for various values of r and fixed λ. Different values of r lead to different critical regimes.



The phase diagram. In the lower (blue) region a system possesses only the 2nd order phase transition; in the region in-between the lines there are both 1st and 2nd order phase transitions at different temperatures; in the upper region (yellow) only 1st order phase transition occurs

NEW RESULTS and OUTLOOK

Phase diagram of the Potts model with invisible states on a scale free network for a given q and different r and λ (after numerical analysis of the free energy (5))



Three regions, presented here, differ in critical behaviour. In the lower (blue) region system possesses only second order phase transition; in the region in-between the lines there are both first and second order phase transitions at different temperatures; in the upper region (yellow) only the first order phase transition occurs.

- Presence of non-integer marginal dimensions is ubiquitous in criticality of complex systems. $(q+r)-$state Potts model is shown to be one of them.
- the number of invisible states can change the universality class of the standard models on complete graph or even on a scale-free network [3], when the degree distribution decay exponent plays a similar role.
- Phase diagram is governed by two marginal dimensions r_{c1} and r_{c2} at different q and λ. q, r, λ play a role of global parameters that influence on the critical behavior of the system.
- The last conclusion may serve as one more example of the 1st order behavior in percolation-like phase transitions.

References

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