Operator expansions, layer susceptibility and correlation functions in boundary CFT

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MECO46: 11-13 May 2021, Riga, Latvia

Conformal invariance in $\mathbb{R}^d$

- short-distance interactions
- translations $x'_\alpha = x_\alpha + a_\alpha$
- rotations $x'_\alpha = b_{\alpha\beta} x_\beta$
- scale transformations $\boldsymbol{x}' = \lambda \boldsymbol{x}$

$\Rightarrow$ Conformal invariance $\boldsymbol{x}' = \lambda(\boldsymbol{x})\,\boldsymbol{x}$

$\overset{SI}{\longrightarrow}$ $O(x') = \lambda^{-\Delta_O} O(x)$ $\quad < O(\boldsymbol{x}) O(\boldsymbol{x}') > = C\,|\boldsymbol{x} - \boldsymbol{x}'|^{-2\Delta_O}$

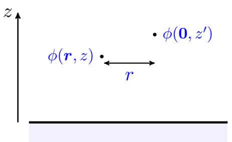
$\overset{CI}{\longrightarrow}$ $O(x') = [\lambda(x)]^{-\Delta_O} O(x)$ $\quad \lambda(x) = |\partial \boldsymbol{x}'/\partial \boldsymbol{x}|^{1/d}$

Special conformal transformations

$$\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^2} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|^2} + \boldsymbol{a}$$

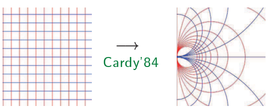
$a \to 0,\ |\boldsymbol{x}' - \boldsymbol{x}| \to 0$: $\quad \boldsymbol{x}' = \lambda(\boldsymbol{x})\,\boldsymbol{x} \quad \lambda(\boldsymbol{x}) = 1 - 2(\boldsymbol{a}\boldsymbol{x}) + O(a^2)$

Semi-infinite systems: $\mathbb{R}^d_+$



- $\phi(\boldsymbol{x})$ — bulk fields
- $\phi(\boldsymbol{r}, 0)$ — boundary fields

$\longleftarrow \equiv \phi(r) \quad \longleftarrow$ BC !

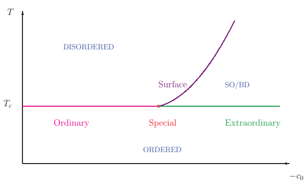


- local fluctuating values $\equiv$ OPERATORS

$O_\Delta = \phi^2,\ \phi^4,\ \phi^2\nabla^2\phi^4 \ldots$

$\hat{O}_{\hat\Delta} = \phi(r), \phi^2(r),\ T_{zz}(r) \ldots$

Phase diagram

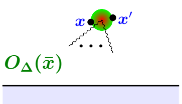


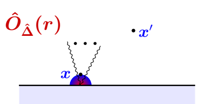
Two-point correlation function of bulk fields

$$G(\boldsymbol{x}, \boldsymbol{x}') \equiv \langle \phi(\boldsymbol{r}_1, z)\phi(\boldsymbol{r}_2, z')\rangle = \frac{g(\xi)}{(4zz')^{\Delta_\phi}}; \qquad g(\xi) \ — \ ?$$

$$\xi = \frac{r^2 + (z' - z)^2}{4zz'} = \frac{|\boldsymbol{x} - \boldsymbol{x}'|^2}{4zz'} \qquad \text{conformal cross-ratio}$$

Short-distance *(operator)* expansions





$\beta = \Delta + 1 - d/2$ conformal blocks $\hat\beta = \hat\Delta + 1 - d/2$

$$\mathcal{G}_{\rm ope}(\Delta;\xi) = \xi^{\Delta/2}{}_2F_1\left(\tfrac{\Delta}{2}, \tfrac{\Delta}{2}; \beta; -\xi\right) \qquad \mathcal{G}_{\rm boe}(\hat\Delta;\xi) = \xi^{-\hat\Delta}{}_2F_1\left(\hat\Delta, \hat\beta; 2\hat\beta; -\xi^{-1}\right)$$

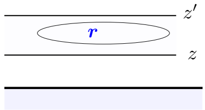
$$g(\xi) = \xi^{-\Delta_\phi} \sum_{\Delta \ge 0} \lambda_\Delta\, \mathcal{G}_{\rm ope}(\Delta;\xi) \ = \ \sum_{\hat\Delta \ge 0} \mu^2_{\hat\Delta}\, \mathcal{G}_{\rm boe}(\hat\Delta;\xi) = g(\xi)$$

bootstrap equation

Correlation function *Radon transformation** Layer suceptibility

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{g(\xi)}{(4zz')^{\Delta_\phi}} \qquad \xi|_{r=0} = \frac{(z' - z)^2}{4zz'} \equiv \rho \qquad \chi(z, z') = (4zz')^{\frac{1-\eta}{2}}\ \hat{g}(\rho)$$

$$\chi(z, z') = \int d^{d-1}r\, G(r; z, z') = \tilde{G}(p = 0; z, z')$$



$$\left.\begin{aligned} \hat{g}(\rho) &= \frac{\pi^\lambda}{\Gamma(\lambda)} \int_0^\infty du\, u^{-1+\lambda} g(u + \rho) \\ g(\xi) &= \frac{\pi^{-\lambda}}{\Gamma(-\lambda)} \int_0^\infty d\rho\, \rho^{-1-\lambda} \hat{g}(\rho + \xi) \end{aligned}\right\} \ \text{Radon transformation} \qquad \lambda = \frac{d-1}{2}$$

*McAvity Osborn95 *CFTs near a boundary in general dimensions* NPB**455** 522

Radon transformation and BOE blocks

Dey Hansen Shpot *Operator expansions, layer susceptibility and two-point functions in BCFT* JHEP **2020** 051

$$G_{con}(r; z, z') = (4zz')^{-\Delta_\phi} g_{con}(\xi) \ \longleftrightarrow\ \chi(z, z') = (4zz')^{\lambda - \Delta_\phi} X(\zeta)$$

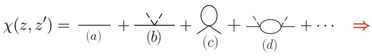
one-to-one correspondence between BOE blocks and ζ powers

$$g_{con}(\xi) = \textstyle\sum_{\hat\Delta > 0} \mu^2_{\hat\Delta}\, \mathcal{G}_{\rm boe}(\hat\Delta;\xi) \quad \overset{R}{\longleftrightarrow} \quad X(\zeta) = \textstyle\sum_{\hat\Delta > 0} c_{\hat\Delta}\, \zeta^{\hat\Delta - \lambda}$$

in scaling functions of the correlator and layer susceptibility

$$\xi = \frac{r^2 + (z' - z)^2}{4zz'} \qquad \lambda = \frac{d-1}{2} \qquad \zeta = \frac{\min(z, z')}{\max(z, z')}$$

An application: EXTRAORDINARY TRANSITION 1Loop: $O(\varepsilon = 4 - d)$



$$\left.\begin{aligned} \chi_L(z, z') &= \sqrt{4zz'}\ \zeta^{\frac{5-\varepsilon}{2}} C_d [1 + \varepsilon\, h(\zeta)] \\ \chi_T(z, z') &= \sqrt{4zz'}\ \zeta^{\frac{3-\varepsilon}{2}} \tilde{C}_{d-1} [1 + \varepsilon\, j(\zeta)] \end{aligned}\right\} \chi(z, z') = (4zz')^{\frac{1-\eta}{2}} \zeta^{\frac{\eta_\parallel - 1}{2}} X(\zeta)$$

$$h(\zeta) = h_0(\zeta) + h_1(\zeta) + h_1(-\zeta)$$

$$h_0(\zeta) = \frac{1}{140(n+8)}\left[203n + 3140 - 10(7n + 96)\zeta^{-2} + 20(7n + 128)\zeta^{-4}\right]$$

$$h_1(\zeta) = \frac{(21n + 204)\,\zeta(1 + \zeta^2) + 4(7n + 74)\,\zeta^2 - 72(1 + \zeta^4)}{42(n+8)\,\zeta^6}(1 - \zeta)^3 \ln(1 - \zeta)$$

★ expand in powers of ζ — get the BOE decomposition ★

NEW explicit results: $g^{\rm con}_{L,T}(\xi) = g^{(0)}_{L,T}(\xi) + \varepsilon\, g^{(1)}_{L,T}(\xi) + O(\varepsilon^2)$

$$\chi_L(z, z') = (4zz')^{\lambda - \Delta_\phi} \zeta^{-\lambda} \sum_{\hat\Delta = d,\, k} c_{\hat\Delta} \zeta^{\hat\Delta} + O(\varepsilon^2) \qquad k = 6, 8, 10, \ldots$$

$$c_d = \frac{1}{10}\left[1 + \frac{76 - n}{60\,(n+8)}\varepsilon\right] \qquad c_k = 2\,\frac{k\,(k-3)\,(n+8) - 2\,(5n+76)}{(k-5)_3(k)_3\,(n+8)}\,\varepsilon$$

$$\overset{R}{\Rightarrow} \quad g_L{}^{\rm con}(\xi) = \sigma_d c_d\, \mathcal{G}_{\rm boe}(4 - \varepsilon;\xi) + \sum_{k=6,8,\ldots}^{\infty} \sigma_k\, c_k\, \mathcal{G}_{\rm boe}(k;\xi) + O(\varepsilon^2)$$

$$g_L^{(0)}(\xi) = \frac{1}{2\xi} - \frac{1}{2(\xi+1)} + 6 + 3(2\xi+1)\ln\frac{\xi}{\xi+1}$$

$$g_L{}^{(1)}(\xi) = \frac{1 + \ln[\xi(\xi+1)]}{4\xi(\xi+1)} + 6\ln\xi + 3(2\xi+1)\ln\xi \cdot \ln\frac{\xi}{\xi+1}$$
$$+ \left[\frac{72\xi(2\xi+3) + n + 80}{4(n+8)}\ln\frac{\xi}{\xi+1} - 2\,\frac{(5n+52)\xi + 2(2n+19)}{n+8}\right]\ln\frac{\xi}{\xi+1}$$
$$+ 2\,\frac{n+14}{n+8}\left[1 + 3(2\xi+1)Li_2\left(-\frac{1}{\xi}\right)\right]$$

O U T L O O K

- Global GOAL: *bootstrap equation for the layer susceptibility*
- HOPE • *match power series instead of ${}_2F_1$ series* ⇒ practical usefulness
- Learn • *use Radon transformation to produce OPE decompositions in simplest cases*
- Non-trivial • *full BOE/OPE decompositions for $d = 2$* ISING — NEW — cf. Liendo Rastelli van Rees JHEP 2013 113
- Apply • *Radon transformation to the "OPE side" of bootstrap equation*
- Try • *BOE/OPE decompositions for ${}_pF_q$ of special arguments in* NUMBER THEORY