Random surface roughening with quenched disorder and turbulent environment

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Kardar-Parisi-Zhang model (1)

Figure 1: Stochastic growth of a flame front and a sand seabed

- Kardar-Parisi-Zhang (KPZ) model is a well-known non-equilibrium dynamics model that describes a wide range of phenomena: random surface roughening, solidification and flame fronts, smoke and colloid aggregates, tumors, etc. [1], [2]. KPZ model is a nonlinear differential equation for the field $h(x) = h(x,t)$ (profile height) that depends on the d-dimensional coordinate x and time t:

$$\partial_t h = \varkappa_0 \partial^2 h + \frac{\lambda_0}{2}(\partial h)^2 + f \qquad (1)$$

Here $\partial_t = \partial/\partial_t$, $\partial_i = \partial/\partial_i$, $\partial^2 = \partial_i\partial_i$, $(\partial h)^2 = (\partial_i h)(\partial_i h)$, $i = 1, ..., d$. Parameter $\varkappa_0 > 0$; λ_0 can be positive or negative, f is random noise.

- Let us put $\lambda_0 = 1$ (this can be achieved by rescaling of the parameters).
- Let us choose f in the form

$$\langle f(x) f(x')\rangle = D_0 \delta^{(d)}(x - x') \qquad (2)$$

This is “spatially quenched” (time-independent) noise; experimental data [3] indicates that it better models landscape erosion than white noise. Here D_0 is a positive amplitude.

Velocity ensemble (2)

- Turbulent motion of the environment is modelled by the Kazantsev–Kraichnan ensemble [4] that has a distribution with zero mean and the correlation function of the form

$$\langle v_i(t,x) v_j(t',x')\rangle = \delta(t-t') D_{ij}(x-x')$$
$$D_{ij}(r) = B_0 \int_{k>m} \frac{dk}{(2\pi)^d k^{d+\xi}} P_{ij}(k) e^{i k_q r_q} \qquad (3)$$

$P_{ij}(k) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, it reflects the incompressibility of the fluid ($\partial_i v_i = 0$), $k = |\mathbf{k}|$ is the wave number, $B_0 > 0$ is a positive amplitude, $0 < \xi < 2$. The cutoff $k > m$ serves as an infrared (IR) regularization.
- The advection by the velocity field is realised by "minimal" replacement $\nabla_t h = \partial_t h + (v_i \partial_i) h$ in Eq. (1).

Field-theoretic reformulation (3)

- Stochastic problem (1) is equivalent to field theory with an increased number of fields $\{h, h', v_i\}$ and action functional $S(\Phi) = S_h(\Phi) + S_v(\Phi)$ [6], where

$$S_h(\Phi) = \frac{1}{2} h' D_0 h' + h'\{-\nabla_t h + \varkappa_0 \partial^2 h + \frac{1}{2}(\partial h)^2\},$$
$$S_v(\Phi) = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}' v_i(t,\mathbf{x}) D^{-1}_{ij}(\mathbf{x}-\mathbf{x}') v_j(t,\mathbf{x}'). \qquad (4)$$

- Here $D^{-1}_{ij}(\mathbf{x}-\mathbf{x}')$ is the kernel of the inverse operator D^{-1}_{ij} for integral operator from Eq. (3); integrations over (x,t) are implied in expression for $S_h(\Phi)$.

Renormalization group analysis (4)

Canonical dimensions analysis shows that:

- New term $\frac{1}{2} u h' vv$ must be added in order for the problem (4) to become renormalizable. There are 5 divergent Green’s functions.
- All the coupling constants $g_0 = D_0/\varkappa_0^4$, $w_0 = B_0/\varkappa_0$, u become simultaneously dimensionless at $d = 4$ (it is a logarithmic dimension), and $\varepsilon = 4 - d$, ξ are the expansion parameters of the perturbation theory.
- Renormalized action functional (Z_i are renormalization constants) is:

$$S_R(\Phi) = \frac{1}{2} Z_1 h' D h' + h'\{-\partial_t h - Z_5 v_i \partial_i h + Z_2 \varkappa \partial^2 h + \frac{1}{2} Z_3 (\partial h)^2\} + S_v + Z_4 \frac{u}{2} h' v^2 \qquad (5)$$

One-loop calculation (5)

- Feynman diagrams technique was used to calculate the diverging 1-irreducible Green’s functions.
- All calculations were made in the leading order of (ε,ξ) expansion. See [5] for example of similar calculations.

Fixed points and and critical dimensions (6)

- The long-time, large-scale asymptotic behaviour of the Green’s functions is determined by the IR attractive fixed points. For field h:

$$\langle h(t,\mathbf{x})\, h(0,\mathbf{0})\rangle \simeq r^{-2\Delta_h} \mathcal{F}\left(t/r^{\Delta_\omega}\right), \qquad (6)$$

where $\mathcal{F}(\dots)$ is a scaling function, Δ_i are critical dimensions.
- RG analysis reveals that there are six fixed points ($x = wu$):
  - $g^* = x^* = w^* = 0$; $\Delta_\omega = 2$, $\Delta_h = 0$, $\Delta_v = 1 - \xi/2$
  - $g^* = -\varepsilon$, $w^* = 0$, $x^* = 0$; $\Delta_\omega = 2 - \varepsilon/4$, $\Delta_h = 0$, $\Delta_v = 1 - \varepsilon/8 - \xi/2$
  - $g^* = -2(\varepsilon + 2\xi)$,$w^* = -4(\varepsilon + 4\xi)/3$, $x^* = w^*$; $\Delta_\omega = 2 + \xi$, $\Delta_h = 0$, $\Delta_v = 1 - \xi$
  - $g^* = -2(\varepsilon - 2\xi)$, $w^* = 4(4\xi - \varepsilon)/3$, $x^* = 4(2\xi - \varepsilon)/3$; $\Delta_\omega = 2 - \xi$, $\Delta_h = 0$, $\Delta_v = 1 - \xi$
  - $g^* = 0$, $w^* = 0$, $x^* = -8\xi/3$; $\Delta_\omega = 2$, $\Delta_h = 0$, $\Delta_v = 1 - \xi/2$
  - $g^* = -\varepsilon$, $w^* = 0$, $x^* = -8\xi/3$; $\Delta_\omega = 2 - \varepsilon/4$, $\Delta_h = 0$, $\Delta_v = 1 - \varepsilon/8 - \xi/2$
- They become attractive at certain values of the parameters (ε, ξ):

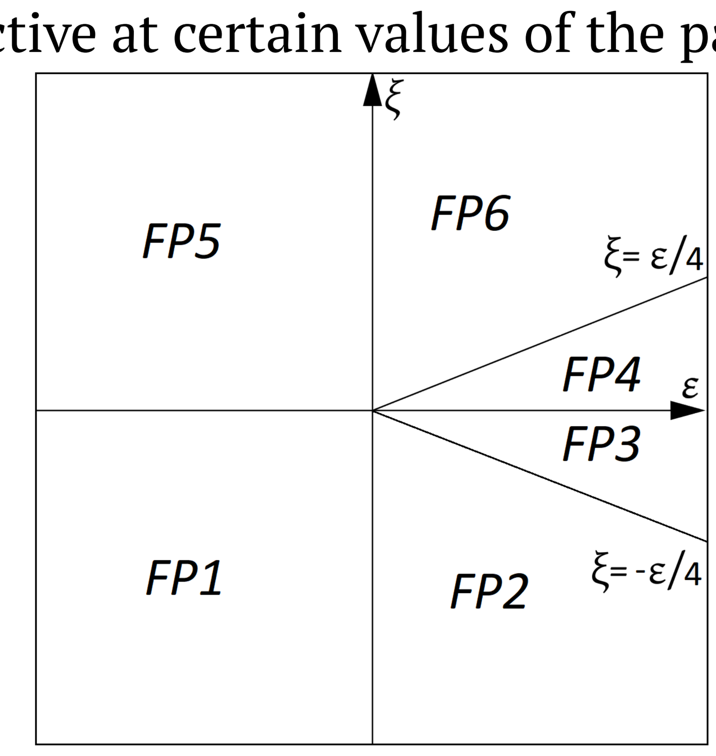

Figure 2: Regions of stability of the fixed points in the model

Conclusion (7)

- KPZ equation with spatially quenched noise advected by velocity field modelled by Kazantsev-Kraichnan ensemble is non-renormalizable theory; a new term $h'vv$ must be added in order to apply renormalization procedure (induced nonlinearity).
- Coordinates of fixed points of RG equations were calculated in one-loop approximation.
- The most realistic values of parameters (d=3, 2, 1, ξ=4/3 – Kolmogorov turbulence) correspond to the fixed point $g^* = -\varepsilon$, $w^* = 0$, $x^* = -8\xi/3$.

References (8)

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