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A new method of solution of the Wetterich equation and its applications

Abstract

A new truncation scheme is proposed [1] to solve the Wetterich exact renormalization group equation [2, 3]

$$\frac{\partial}{\partial k}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left\{\left[\Gamma_k^{(2)}[\phi] + R_k\right]^{-1}\frac{\partial}{\partial k}R_k\right\} \qquad (1)$$

for the the effective average action $\Gamma_k[\phi]$, depending on the infrared cut-off scale k. The natural domain of validity of the derivative expansion appears to be limited to small values of normalized wave vectors q/k. To the contrary, the new approximation scheme has the advantage to be valid for any q/k, therefore, it can be auspicious in many current and potential applications of the celebrated Wetterich equation and similar models. In distinction from the derivative expansion, derivatives are not truncated at a finite order in the new scheme. The derivative expansion up to the ∂^2 order is just the small-q approximation of our new equations at the first order of truncation. The RG flow equations at this order are derived and approximately solved as an example. Furthermore, a new method of functional truncations is tested for such a solution.

The Wetterich equation

In the Wetterich equation (1), the average effective action $\Gamma_k[\phi]$ depends on the averaged order parameter $\phi(\mathbf{x})$ with components $\phi^j(\mathbf{x}) = \langle\chi^j(\mathbf{x})\rangle$, $j = 1, \ldots, N$. Here, the averaging of the original order parameter χ is performed in the presence of external field $J(\mathbf{x})$, so that ϕ is determined by J. The effective action contains a smooth infrared (lower) cut-off of fluctuations with wave vector magnitude $q \lesssim k$, represented by the term R_k in (1). The upper cut-off at $q = \Lambda$ is also included. In the wave-vector space, the quantity $\Gamma_k^{(2)}[\phi]$ is a matrix with elements

$$\left(\Gamma_k^{(2)}\right)_{ij}(\mathbf{q}, \mathbf{q}') = \frac{\delta^2\Gamma_k[\phi]}{\delta\phi^i(-\mathbf{q})\,\delta\phi^j(\mathbf{q}')}\,. \qquad (2)$$

The cut-off R_k is a diagonal matrix with elements $R_{k,ij}(\mathbf{q}, \mathbf{q}') = R_k(q)\,\delta_{ij}\,\delta_{\mathbf{q},\mathbf{q}'}$, $R_k(q) = (\alpha Z_k q^2)/(e^{q^2/k^2} - 1)$, where Z_k is a renormalization constant and α is an optimization parameter. Derivatives of $\Gamma_k[\phi]$ are related to correlation functions.

The new truncation scheme

We consider the effective action of the $O(N)$ symmetric model at $N = 1$ in the following general form

$$\Gamma_k[\phi] = \int\Bigg(U_k(\rho(\mathbf{x})) + V^{-1}\sum_{\mathbf{q}_1,\mathbf{q}_2}\left[\theta_k^{(1)}(\rho(\mathbf{x});\mathbf{q}_1) + \theta_k^{(2)}(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2)\right]\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)e^{i(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}$$
$$+V^{-2}\sum_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}\left[\theta_k^{(3)}(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) + \theta_k^{(4)}(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)\right]\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(\mathbf{q}_4)$$
$$\times e^{i(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)\mathbf{x}} + \cdots\Bigg)d\mathbf{x}\,, \qquad (3)$$

where $\rho = \phi^2/2$, and $\theta_k^{(m)}(\rho(\mathbf{x});\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_m) = 0$, if $\mathbf{q}_j = 0$ holds for any of $j \in [1, m]$. The term with $\theta_k^{(m)}$ includes in a closed form all relevant (corresponding to the symmetry of the model) terms of the kind $\phi^\ell\,\frac{\partial^{\alpha_1}\phi}{\partial\mathbf{x}^{\alpha_1}}\frac{\partial^{\alpha_2}\phi}{\partial\mathbf{x}^{\alpha_2}}\cdots\frac{\partial^{\alpha_m}\phi}{\partial\mathbf{x}^{\alpha_m}}$, where $\alpha_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jd})$ is the multi-index in the standard notations of the functional analysis. The derivative expansion is recovered from (3) by expanding the functions $\theta_k^{(m)}$ in small q_j limit. In (3), however, the magnitudes of wave vectors q_j need not to be small.

We propose an approximation scheme, where terms $\theta_k^{(j)}$ with $j \leq m$ are included in the m-th approximation. It is a truncation. However, at any m, the order of derivatives included is not limited. This is a salient difference from a truncated derivative expansion.

At $m = 1$, we denote $\theta_k^{(1)}(\rho;\mathbf{q}) = \theta_k(\rho;\mathbf{q})$ and use a transformed variable $\Psi_k(\rho;\mathbf{q}) = \theta_k(\rho;\mathbf{q}) + 2\rho\,\theta_k'(\rho;\mathbf{q})$. Furthermore, we write our equations in a dimensionless form with $\Psi_k(\rho;\mathbf{q}) = \frac{1}{2}Z_k q^2 f_k(\tilde\rho;y)$, where $y = q^2/k^2$, $\tilde\rho = Z_k k^{2-d}\rho$ and $Z_k = \lim_{q\to 0}\left(\frac{2}{q^2}\Psi_k(0;\mathbf{q})\right)$, as well as $U_k(\rho) = k^d u_k(\tilde\rho)$ and $R_k(q) = Z_k q^2 r(y)$. We consider the running exponent $\eta(k) = -\frac{d}{dt}\ln Z_k$, where $t = \ln\left(\frac{k}{\Lambda}\right)$.

The above transformations lead to the following RG flow equations:

$$\frac{\partial u_k(\tilde\rho)}{\partial t} = -d\,u_k(\tilde\rho) + (d-2+\eta(k))\tilde\rho\,u_k'(\tilde\rho) - \frac{K_d}{4}\int_0^{\Lambda^2/k^2}\frac{y^{\frac{d}{2}-1}\zeta_k(y)dy}{\mathcal{P}_k(\tilde\rho,y)}\,, \qquad (4)$$

$$\frac{\partial f_k(\tilde\rho;y)}{\partial t} = \eta(k)f_k(\tilde\rho;y) + \tilde\rho(d-2+\eta(k))f_k'(\tilde\rho;y) + 2y\frac{\partial f_k(\tilde\rho;y)}{\partial y}$$
$$+\frac{K_d}{4}\left(f_k'(\tilde\rho;y) + 2\tilde\rho f_k''(\tilde\rho;y)\right)\int_0^{\Lambda^2/k^2}\frac{y_1^{\frac{d}{2}-1}\zeta_k(y_1)dy_1}{\mathcal{P}_k^2(\tilde\rho,y_1)} - y^{-1}\left[\hat C_k(\tilde\rho,y) - \hat C_k(\tilde\rho,0)\right]\,, \qquad (5)$$

where $\zeta_k(y) = 2y^2r'(y) + \eta(k)\,yr(y)$, $w_k(\tilde\rho) = u_k'(\tilde\rho) + 2\tilde\rho\,u_k''(\tilde\rho)$, $\mathcal{P}_k(\tilde\rho,y) = w_k(\tilde\rho) + y[f_k(\tilde\rho;y) + r(y)]$ and

$$\hat C_k(\tilde\rho,y) = \tilde\rho\,\tilde K_d\int_0^{\Lambda^2/k^2}\int_0^{\pi}\zeta_k(y_1)\,\Theta\left(\frac{\Lambda^2}{k^2} - Y\right)y_1^{\frac{d}{2}-1}(\sin\theta)^{d-2}$$
$$\times\frac{\left(w_k'(\tilde\rho) + \frac{1}{2}\left[yf_k'(\tilde\rho,y) + y_1f_k'(\tilde\rho,y_1) + Yf_k'(\tilde\rho,Y)\right]\right)^2}{\mathcal{P}_k(\tilde\rho,Y)\,\mathcal{P}_k^2(\tilde\rho,y_1)}\,dy_1d\theta\,, \qquad (6)$$

where $Y = y + y_1 + 2\sqrt{yy_1}\cos\theta$ and $\tilde K_d = S(d-1)/(2\pi)^d$, where $S(d)$ is the area of the unit sphere in d dimensions.

Approximate solution and functional truncations

In [1], a simple polynomial approximation $u_k(\tilde\rho) \approx u_{0,k} + u_{1,k}\,\tilde\rho + u_{2,k}\,\tilde\rho^2$, and $f_k(\tilde\rho;y) \approx f_{0,k}(y) + f_{1,k}(y)\,\tilde\rho$ has been considered, showing that reasonable results can be obtained from our equations. Similar functional truncations within the derivative expansion have been used, e. g., in [4], with more terms included and $\tilde\rho - \tilde\rho_0$ instead of $\tilde\rho$, where $\tilde\rho_0$ corresponds to the minimum of the potential $u(\tilde\rho)$. It provided estimates for critical exponents ν and η, those for the correction-to scaling exponent ω being not reported. The latter ones have been provided by a purely numerical solution on a grid in [5].

Recently, we have tested a new expansion:

$$u_k(\tilde\rho) - u_k(0) = (1-s)^{-\mu}\left(u_{1,k}\,s + u_{2,k}\,s^2 + \ldots\right)\,, \qquad (7)$$
$$f_k(\tilde\rho;y) = f_{0,k}(y) + f_{1,k}(y)\,s + f_{2,k}(y)\,s^2 + \ldots \qquad (8)$$

where $s = \tilde\rho/(\tilde\rho + \tilde\rho_0)$, $\mu = d/(d-2+\eta)$ and $\tilde\rho_0$ is a constant – an optimization parameter. It allows to interpolate between the expansion in powers of $\tilde\rho$ at $\tilde\rho \to 0$ and the large-$\tilde\rho$ asymptotic. Eventually, this method can give good estimates for ν, η and also ω. We have performed preliminary tests within the local potential approximation (LPA) (where $f_k(\tilde\rho;y) \equiv 1$) in three dimensions - see Fig. 1.

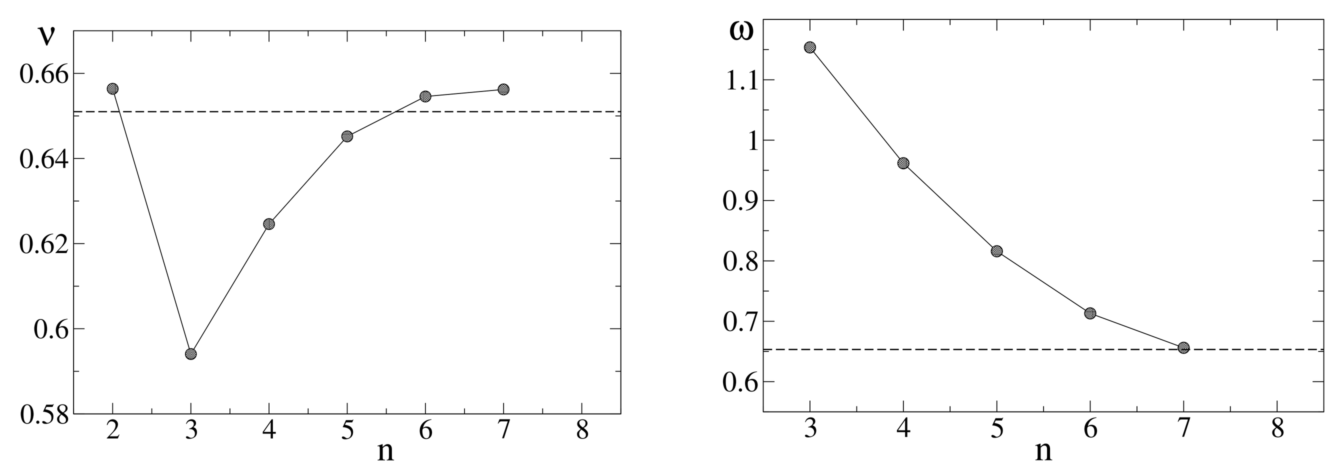
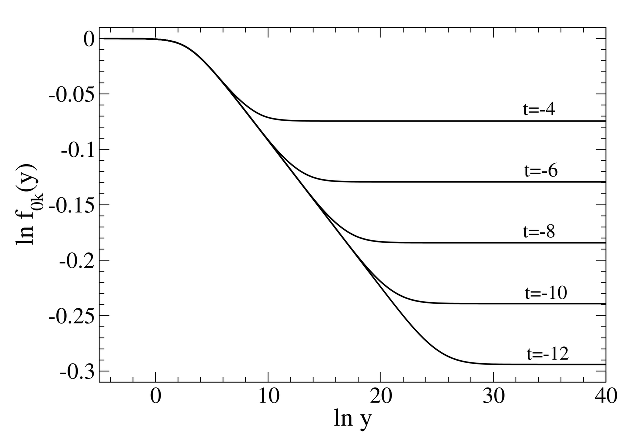


FIGURE 1: The values of ν (left) and ω (right) obtained within LPA depending on the truncation order n in (7). The dashed lines indicate numerical values reported in [5]. Not yet optimized values of $\alpha = 2$ and $\tilde\rho_0 = 0.5$ are used.

Testing beyond the LPA is in progress. It is already clear that the new expansion provides better results for ν and η than the expansion in powers of $\tilde\rho$ at low orders of truncation. We have calculated also the scaling function $f_k(\tilde\rho;y)$, related to the two-point correlation function:



The critical scaling function $f_k(\tilde\rho;y)$ at different renormalization scales t. A smooth upper cut-off, related to the large-y crossover, has been used for convenience. The crossover at $y \sim 1$ is induced by the infrared cut-off.

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