Lagrangian description of dissipative oscillator

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Introduction

Dissipation is a statistical process caused by the coupling between the system and its environment. The equation of motion for such a process cannot be derived from a Lagrangian (because the equation contains a non-self-adjoint differential operator) if only the degrees of freedom corresponding to the system are considered. If more degrees of freedom are added, it is possible to obtain the correct equations from a Lagrangian. One way of doing so is by including the environment[1, 2] into the problem, which is a physics motivated idea. Another way, is to introduce some abstract mathematical function, for example, a potential which generates the physical observable[3, 4, 5, 6]. This is more of a mathematical workaround and only doubles the degrees of freedom instead of adding infinitely many of them. Also, this approach eliminates the problem of choosing a proper model for the environment.

Formulating the theory

Consider the dissipative process described by $\mathcal{D}u(t) = c(t)$, where $\mathcal{D}$ is a non-self-adjoint linear differential operator. The potential that generates the measurable can be defined using $u(t) = \tilde{\mathcal{D}}\phi(t)$, where $\tilde{\mathcal{D}}$ is the adjoint of $\mathcal{D}$. For $\phi(t)$, the Lagrangian and the Euler–Lagrange equations are

Lagrangian and equation of motion for the potential

$$L = \frac{1}{2}\big(\tilde{\mathcal{D}}\phi(t)\big) \cdot \big(\tilde{\mathcal{D}}\phi(t)\big) - \phi(t) \cdot c(t),$$
$$0 = \mathcal{D}\big(\tilde{\mathcal{D}}\phi(t)\big) - c(t).$$

As the method is based on doubling the degrees of freedom, it can be assumed that the solution may contain non-physical terms. Indeed, if one separates the solution $\phi(t) = \varphi(t) + \lambda(t)$ where $\varphi(t) \in \mathrm{Im}(\tilde{\mathcal{D}})$, and $\lambda(t) \in \mathrm{Ker}(\tilde{\mathcal{D}})$, then calculates the measurable quantity, it is clear that $\varphi(t)$ must contain all information and $\lambda(t)$ must contain none. Usually the non-physical terms are exponentially increasing.

At the level of potentials, only half of the initial conditions are provided by the original physical problem, the other half can be chosen freely. The benefit of this freedom is that the initial conditions can be chosen in a way that ensures the vanishing of non-physical terms. This choice is unique and exists, however highly non-trivial in general. There is no systematic way to perform this task at the moment, only if the analytical solution is known, so numerically solvable equations are unstable.

Potential for the damped harmonic oscillator

The damped harmonic oscillator is the most elementary classical mechanical system that shows dissipative behaviour. It is also the perfect model to demonstrate the essence of this method.

Smart choice of initial conditions

$$\mathcal{D}u(t) = c(t) \;\rightarrow\; \ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0$$
$$\tilde{\mathcal{D}}\phi(t) = u(t) \;\rightarrow\; \ddot{\phi} - 2\lambda\dot{\phi} + \omega^2\phi = x$$

$$L = \frac{1}{2}\left(\ddot{\phi} - 2\lambda\dot{\phi} + \omega^2\phi\right)^2 \qquad (1)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}}{\mathrm{d}t} + \omega^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 2\lambda\frac{\mathrm{d}}{\mathrm{d}t} + \omega^2\right)\phi = 0 \qquad (2)$$

$$\phi(t) = a_1\mathrm{e}^{-(\lambda+\gamma)t} + a_2\mathrm{e}^{-(\lambda-\gamma)t} + b_1\mathrm{e}^{(\lambda+\gamma)t} + b_2\mathrm{e}^{(\lambda-\gamma)t}, \qquad (3)$$

where $\gamma = \sqrt{\lambda^2 - \omega^2}$

Smart choice of initial conditions

$$\phi(0) = \frac{2\lambda x_0 + v_0}{4\lambda(\lambda^2 - \gamma^2)} \qquad a_1 = \frac{(\gamma - \lambda)x_0 - v_0}{8\gamma\lambda(\lambda + \gamma)}$$
$$\dot{\phi}(0) = -\frac{x_0}{4\lambda} \qquad a_2 = \frac{(\gamma + \lambda)x_0 + v_0}{8\gamma\lambda(\lambda - \gamma)}$$
$$\ddot{\phi}(0) = -\frac{v_0}{4\lambda} \qquad b_1 = 0$$
$$\dddot{\phi}(0) = \frac{(\lambda^2 - \gamma^2)x_0 + 2\lambda v_0}{4\lambda} \qquad b_2 = 0$$

Lagrangian of telegrapher's heat conduction

Although the damped oscillator seems redundant, it appears in many models. There are many systems that can be transformed into a damped oscillator problem. One such equation is the telegrapher's heat transport (a la Maxwell–Cattaneo–Vernotte). Using a spatial Fourier-transform, the problem reduces to the dissipative harmonic oscillator.

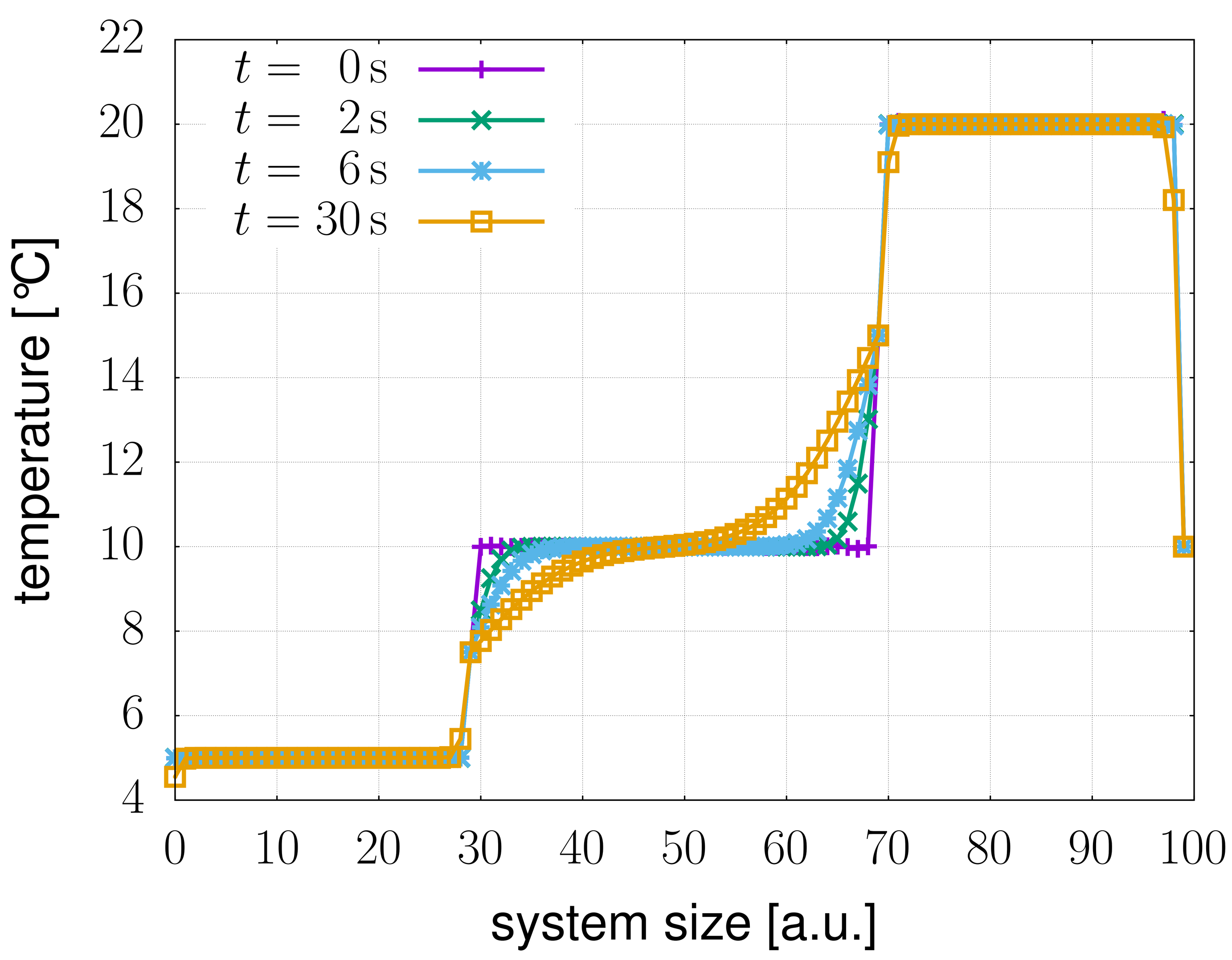

Figure 1: *We simulated heat conduction in a one dimensional silicon bar in the range of* $30 - 70$*, brought into contact with different temperature media on its left and right. The temperature distribution is plotted at different times.*

Outlook

It is important to keep in mind that the main goal of trying to provide an easy-to-use Lagrangian description of dissipative systems is that it seems beneficial to take advantage of the tools coming along with the Lagrangian framework. These tools include the ease of coupling systems, utilizing symmetries, and the possibility of quantization.

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