How does homophily shape the topology of a dynamic network?

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Our dynamical network model

At each step, select an agent \( i \) randomly, then
- If \( k_i > \kappa \), choose a neighbour of \( i \) and delete the link with it with probability \( (1 - J\sigma_i \sigma_j^*)/2 \).
- If \( k_i < \kappa \), choose a non-neighbour of \( i \) and add a link with it with probability \( (1 + J\sigma_i \sigma_j^*)/2 \), where \( \kappa \) represents the preferred degree, set as a half-integer, \( \kappa_i, \sigma_i = \pm 1 \) are the degree and opinion of agent \( i \) respectively. \( J \in [-1,1] \) is to characterize the level of homophily.

Quantities of interest

- The degree distribution: \( p_\kappa \equiv \{ \delta(k - k_i) \delta(\sigma - \sigma_i) \} \), where \( \delta \) is the Kronecker delta function;
- The average degree and variance: \( \bar{\kappa} \equiv \sum_k k p_\kappa(k) \) and \( \nu_\kappa \equiv \sum_k k^2 p_\kappa(k) - \bar{\kappa}^2 \);
- The fraction of CLs: \( \rho \equiv L_{CL}/L \), where \( L \) is the total number of connections, and \( L_{CL} \) is the total number of CLs.

Techniques

Suppose there are \( N \) agents overall, and the fraction of "add" is \( \alpha \).

\[
\begin{align*}
\text{Nodes of degree less than } \kappa & : \rho_\kappa \\
\text{Nodes of degree greater than } \kappa & : \rho_{\kappa'} \qquad \kappa' = 2 \kappa
\end{align*}
\]

Assumption: the population of the left set would fluctuate around \( \alpha^2 N \), while the population of the right set would fluctuate around \((1 - \alpha^2)N\).

Mean-field analysis

Denote by \( L_{CL} \) and \( L_{IL} \) the number of CLs and ILs, respectively. The rate for \( L_{CL} \) to increase is \( \alpha \cdot \frac{1}{2} - \frac{1}{2} \rho \), while for \( L_{IL} \) to decrease is \((1 - \alpha) \cdot \frac{1}{2} - \frac{1}{2} \rho \). In the steady state, balancing these contributions leads to

\[
\alpha(1 - J) = (1 - \alpha)(1 + J) \rho,
\]

where \( \rho := \sum_i \sum_j \delta(k_i (1 - \sigma_i \sigma_j^*)/2) \), i.e., the fraction of CLs. \( (\sigma_{ij}) \) is the adjacency matrix of the network.

Similarly, by balancing the increase and decrease of \( L_{IL} \), we have

\[
\alpha(1 + J) = (1 - \alpha)(1 - J)(1 - \rho).
\]

Solving the above equations, we find the mean-field predictions for the steady state values:

\[
\alpha = \frac{1}{2} (1 - J^2), \quad \rho = \frac{1}{2} \left( 1 + J^2 \right).
\]

Stationary degree distribution

Denote by \( p(k, t) \) the fraction of nodes of degree \( k \) in the network. Then we have the following mean-field equations for \( p(k, t) \). If \( R^0(k) \) and \( R^e(k) \) are the rates at which a node of degree \( k \) adds or cuts a link, then \( p(k, t) \) obeys

\[
\frac{dp(k,t)}{dt} = R^0(k)(1-p(k-1,t)) + R^e(k)(1-p(k+1,t)) - [R^0(k) + R^e(k)]p(k,t),
\]

where \( R^0 \) and \( R^e \) are the rates of adding and removing links, respectively. We can find the rates

\[
R^e = \frac{1}{2} [H(k - \kappa) - \kappa], \quad R^0 = \chi [H(k - \kappa) + (1 - \alpha)],
\]

where \( \chi := \frac{1}{2} (1 - J) (1 - \rho) + \frac{1}{2} (1 + J) \rho \) and \( H \) is the Heaviside step function. We obtain the stationary degree distribution as

\[
p(k) = \begin{cases} 
1 + J^2, & k = \kappa, \\
\frac{1}{2} [(1 + J^2) - k], & k > \kappa, \\
\frac{1}{2} [(1 + J^2) - k], & k < \kappa.
\end{cases}
\]

Average and variance of Degrees

Now the average and the variance of degrees can be obtained directly from \( p(k) \)

\[
\mu = \bar{k} = \kappa_1 + \kappa_2 = \frac{3J^2}{2},
\]

\[
\nu = \nu_\kappa = \left( \kappa_1^2 + \kappa_2^2 + \kappa_1 \kappa_2 + 2 \kappa_1 J^2 + 3 \right)/2,
\]

\[
V = \left( \kappa_1^2 - \kappa_2^2 \right) = (7 + J^4)/4,
\]

where \( \kappa_1 = \lfloor \kappa \rfloor \) and \( \kappa_2 = \lceil \kappa \rceil \).

Future work

- Study the case when \( \mu \neq 0 \), i.e., \( N_{-} \neq N_{+} \);
- Characterizing the phase transition when \( \mu \neq 0 \);
- ...