How does homophily shape the topology of a dynamic network?

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Our dynamical network model

At each step, select an agent i randomly, then

- if $k_i > \kappa$, choose a neighbour of i and delete the link with it with probability $(1 - J\sigma_i\sigma_j)/2$,
- if $k_i < \kappa$, choose a non-neighbour of i and add a link with it with probability $(1 + J\sigma_i\sigma_j)/2$.

where κ represents the preferred degree, set as a half-integer; $k_i, \sigma_i (\sigma_i = \pm 1)$ are the degree and opinion of agent i respectively. $J \in [-1, 1]$ is to characterize the level of homophily.

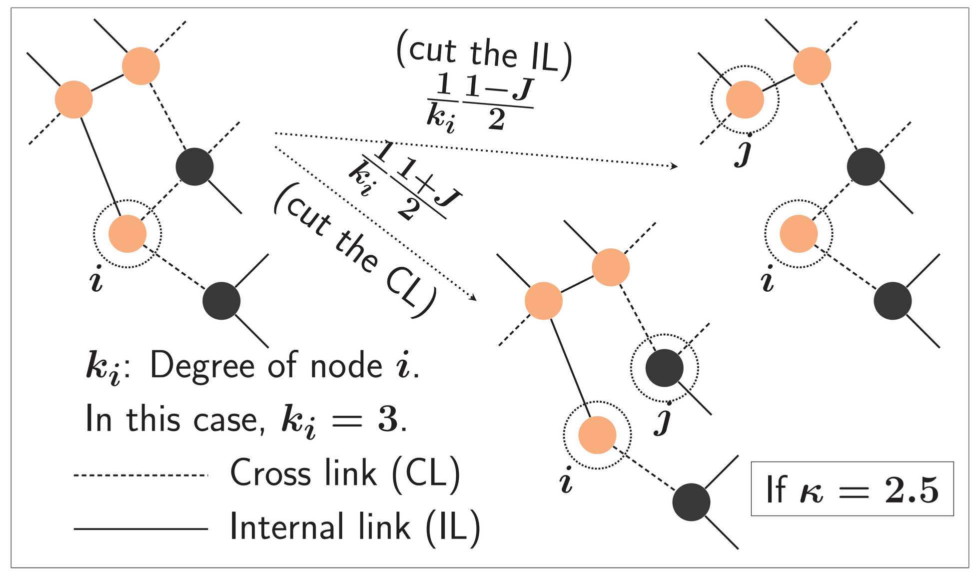


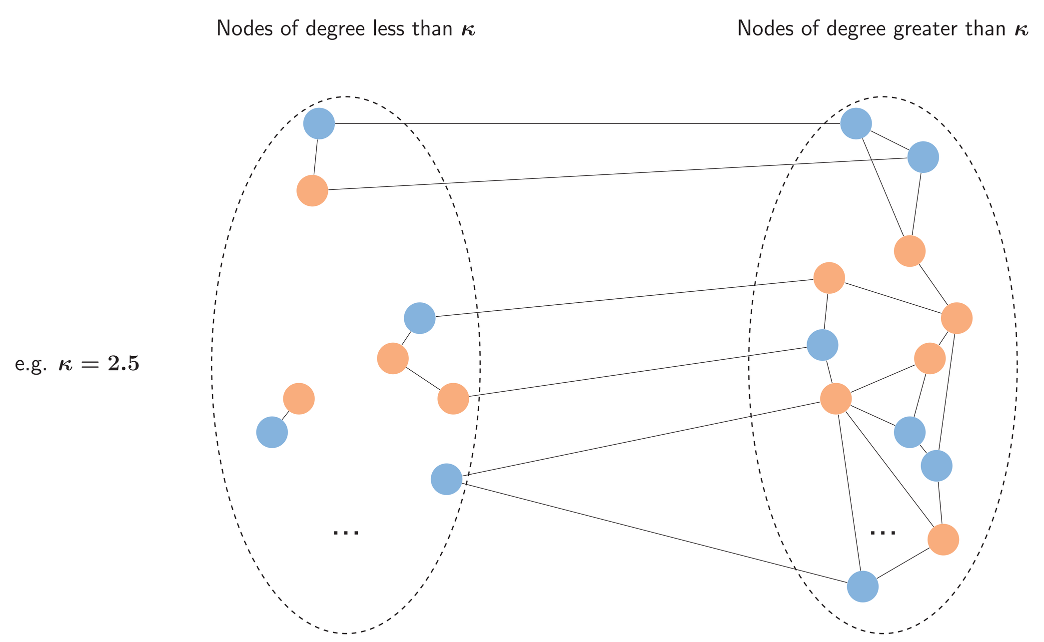
Figure: Illustration of the evolution rule

Quantities of interest

- The degree distribution: $p_\sigma \equiv \langle \delta(k - k_i)\delta(\sigma - \sigma_i)\rangle$, where δ is the Kronecker delta function;
- The average degree and variance: $\mu_\sigma \equiv \sum_k kp_\sigma(k)$ and $V_\sigma \equiv \sum_k k^2 p_\sigma - \mu_\sigma^2$;
- The fraction of CLs: $\rho \equiv L_\times / L$, where L is the total number of connections, and $L_\times$ is the total number of CLs.

Techniques

Suppose there are N agents overall, and the fraction of "adders" is α.



Assumption: the population of the left set would fluctuate around $\alpha^* N$, while the population of the right set would fluctuate around $(1 - \alpha^*)N$.

Mean-field analysis

Denote by $L_\times$ and $L_\odot$ the number of CLs and ILs, respectively. The rate for $L_\times$ to increase is $\alpha \cdot \frac{1}{2} \cdot \frac{1-J}{2}$, while for $L_\times$ to decrease is $(1 - \alpha) \cdot \frac{1+J}{2} \cdot \rho$. In the steady state, balancing these contributions leads to

$$\alpha(1 - J) = 2(1 - \alpha)(1 + J)\rho,$$

where $\rho := \frac{\sum_{i,j=1}^{N} a_{ij}(1-\sigma_i\sigma_j)/2}{2N}$, i.e., the fraction of CLs. (a_{ij}) is the adjacency matrix of the network.
Similarly, by balancing the increase and decrease of $L_\odot$, we have

$$\alpha(1 + J) = 2(1 - \alpha)(1 - J)(1 - \rho).$$

Solving the above equations, we find the mean-field predictions for the steady state values:

$$\alpha = \frac{1}{2}(1 - J^2), \quad \rho = \frac{1}{2} - \frac{J}{1 + J^2}.$$

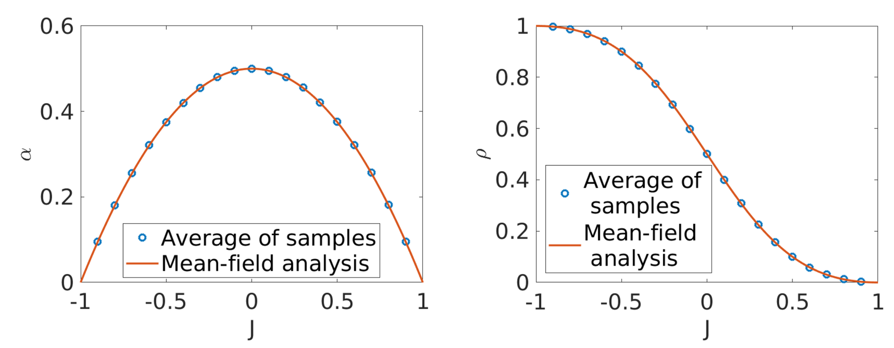


Figure: α and ρ in the case $N = 1000, m = 0$ and $\kappa = 4.5$. Lines are from the above equations and symbols are from simulations, average of data after 100 MCS.

Stationary degree distribution

Denote by $p(k, t)$ the fraction of nodes of degree k in the network. Then we have the following mean-field equations for $p(k, t)$. If $R^a(k)$ and $R^c(k)$ are the rates at which a node of degree k adds or cuts a link, then $p(k, t)$ obeys

$$\frac{dp(k,t)}{dt} = R^a(k-1)p(k-1,t) + R^c(k+1)p(k+1,t) - [R^a(k) + R^c(k)]p(k,t).$$

We can find the rates

$$R^a \simeq \frac{1}{2}[H(\kappa - k) + \alpha], \quad R^c \simeq \chi[H(k - \kappa) + (1 - \alpha)],$$

where $\chi := \frac{1}{2}(1 - J)(1 - \rho) + \frac{1}{2}(1 + J)\rho$ and H is the Heaviside step function.
We obtain the stationary degree distribution as

$$p(k) = \begin{cases} \left(\frac{1+J^2}{3+J^2}\right)^{k - \lfloor\kappa\rfloor} & \text{for } k > \kappa, \\ \left(\frac{1-J^2}{3-J^2}\right)^{\lceil\kappa\rceil - k} & \text{for } k < \kappa. \end{cases}$$

Simulation results

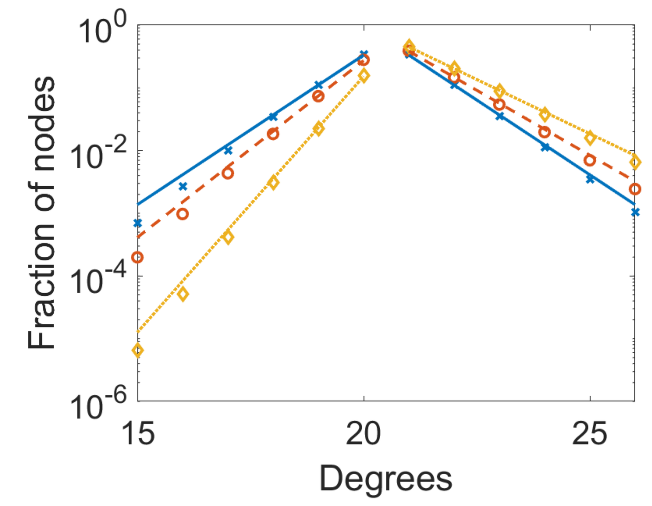


Figure: Degree distribution $p(k)$ vs. degree k for $\kappa = 20.5$ and $J = 0$ (blue), $J = 0.5$ (red) and $J = 0.8$ (yellow).

Average and variance of Degrees

Now the average and the variance of degrees can be obtained directly from $p(k)$

$$\mu = \langle k\rangle = \frac{\kappa_1 + \kappa_2}{2} + \frac{3J^2}{2},$$
$$\langle k^2\rangle = (\kappa_1^2 + \kappa_2^2 + 3J^2(\kappa_1 + \kappa_2) + 5J^4 + 3)/2,$$
$$V = \langle k^2\rangle - \langle k\rangle^2 = (7 + J^4)/4,$$

where $\kappa_1 = \lfloor\kappa\rfloor$ and $\kappa_1 = \lceil\kappa\rceil$.

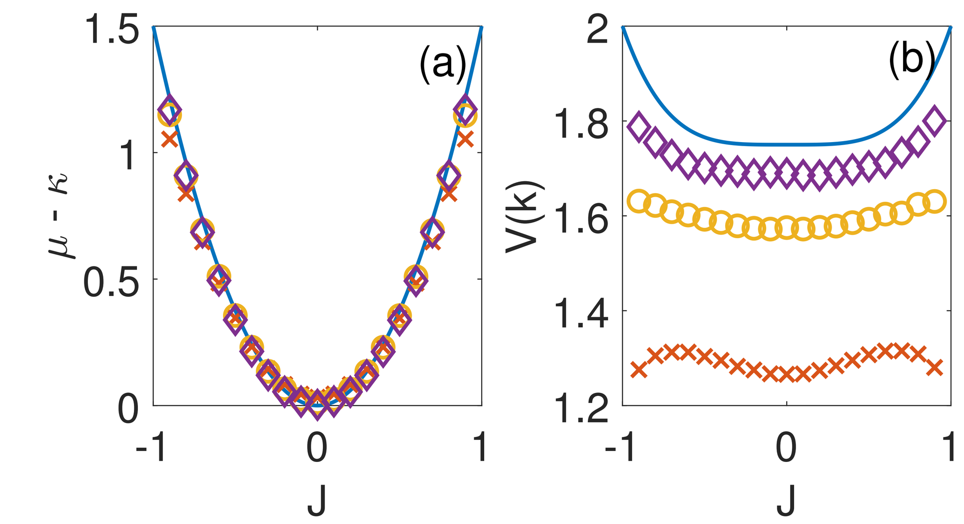


Figure: $\mu - \kappa$ and $V(k)$ vs J for different preferred degree: $\kappa = 5.5$ ($\times$), 20.5 ($\circ$), and 70.5 ($\diamond$). Lines are the average and variance of (??); markers are from simulations. Data are collected after 2000 MCS.

Future work

- Study the case when $m \neq 0$, i.e., $N_+ \neq N_-$;
- Characterizing the phase transition when $m \neq 0$;
- $\cdots$