Heterogeneous excitable systems exhibit Griffiths phases below hybrid phase transitions

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ABSTRACT

In d > 2 dimensional, homogeneous threshold models discontinuous transition occur, but the mean-field solution provides 1/t power-law activity decay and other power-laws, thus it is called mixed-order or hybrid type. It has recently been shown that the introduction of quenched disorder rounds the discontinuity and second order phase transition and Griffiths phases appear. Here we provide numerical evidence, that even in case of high graph dimensional hierarchical modular networks the Griffiths phase of the K = 2 threshold model is present below the hybrid phase transition. This is due to the fragmentation of the activity propagation by modules, which are connected via single links. This provides a widespread mechanism in case of threshold type of heterogeneous systems, modeling the brain or epidemics for the occurrence of dynamical criticality in extended Griffiths phase parameter spaces. We investigate this in synthetic modular networks with and without inhibitory links as well as in the presence of refractory states.

HIERARCHICAL MODULAR NETWORKS

Scheme of network construction

The network was generated beginning with the highest level and adding modules to the next lower level with random connectivity within modules. Lowest level got extra short links to provide single connectedness. Edge density increases from top to bottom levels. Nodes are connected in a hierarchical modular way as if they were embedded in a regular two-dimensional lattice (HMN2d). Due to this construction the Euclidean distance R obeys the relation $p(R) \sim R^{-s}$

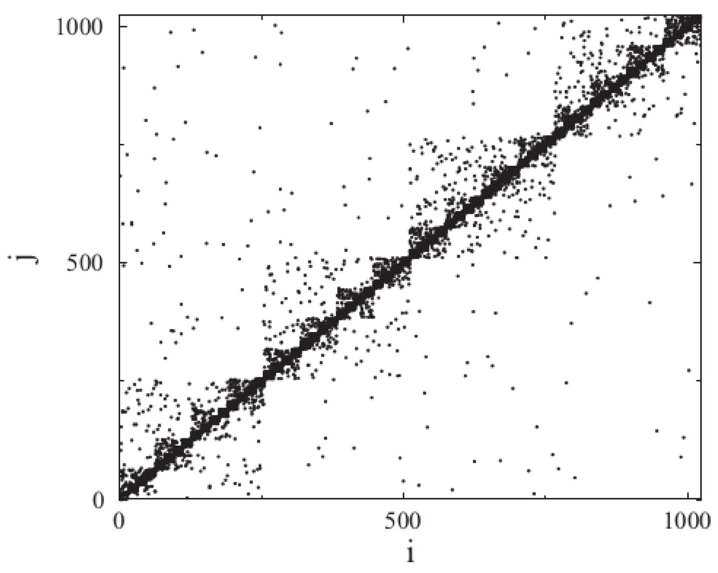

Fig. 1: Plot of the adjacency matrix of an $N = 1024$-sized sample of the HMN2d graph. Black dots denote connections between nodes i and j. The four-level structure is clearly visible in the blocks along the diagonal; additional long-range edges are scattered points around it.

- Fixed average node degree: $\langle k \rangle = 12$
- Number of nodes in a level l: $N_l = 4^{l+1}, l = 0, 1, ..., l_{max}$

Network Properties

- Topological dimension d is defined by $N(r) \sim r^d$
- Small-world type with $d \sim 4.18(5)$. In finite d we expect relevant rare regions, which flip very slowly, causing **non-universal, dynamical scaling around the critical point, called Griffiths Phase**

THRESHOLD MODEL

Two-state system: $x_i = 0, 1$ (active,inactive)

- Conditional activation rule: $\sum_j x_j w_{i,j} \geq K$.
  If this is true
  - nodes become active with activation probability λ

  Otherwise,
  - Nodes become inactive with deactivation probability $\nu = 1 - \lambda$
- Mean-field approximation: probability of site activation ρ and two active neighboring sites can occur in a $(N-1)(N-2)/2$ way. In case of a global acceptance Λ, the creation rate is
$$\frac{1}{2}(N-1)(N-2)\Lambda\rho^2(1-\rho)$$
- Calling $\lambda = (N-1)(N-2)\Lambda/2$, for a full graph of N nodes the rate equation is
$$\frac{d\rho}{dt} = \lambda\,\rho^2(1-\rho) - \nu\,\rho$$
- Real and positive solution: $\Lambda_c = \frac{8}{5(N-1)(N-2)}$
- At the transition point: $\rho(t) - \rho_c \sim t^{-1}$

Power-law (PL) behaviour in a discontinuous transition

MEASUREMENTS

- Density of active nodes $\rho(t) = 1/N \sum_{i=1}^{N} x_i$
- A single pair of active nodes can trigger an avalanche of duration T and spatiotemporal size $s = \sum_{i=1}^{N}\sum_{t=1}^{T} x_i$. It allows us to compute:
  - **probability density functions of avalanche sizes** $p(s)$
  - **final survival time distributions** $p(t)$

RESULTS FOR K=2

Excitatory Model: Weight of links: $w_{ij} = 1$

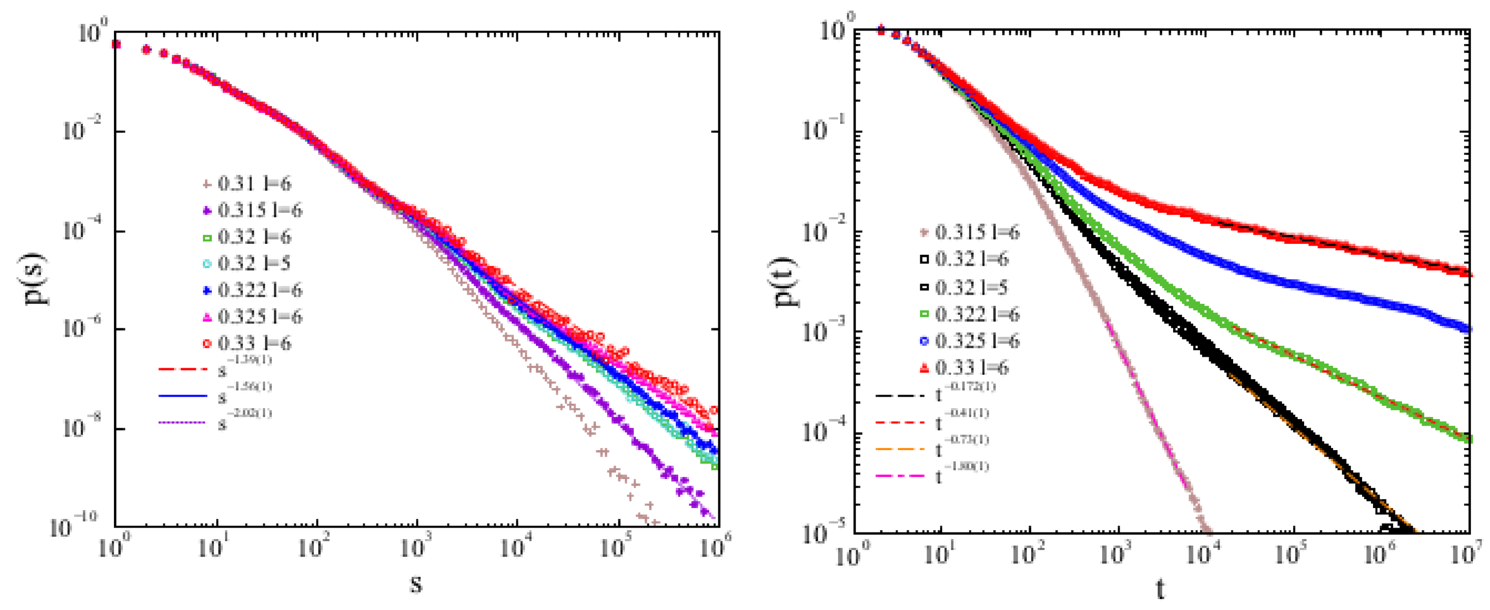

Fig. 2: Left: Avalanche size distributions. Dashed lines show PL fits for the tails: $s > 1000$ at λ = 0.315, 0.322, 0.33. Right: survival probability of the activity. Dashed lines are PL fits for the tails of $\lambda = 0.505, 0.52$ curves. Right: survival probability of the activity. Dashed lines show PL fits for the tails: $s > 10^4$ at $\lambda = 0.315, 0.32, 0.322, 0.33$.

Inhibitory Model: $w_{ij} = 1$ or -1 with probability 0.2

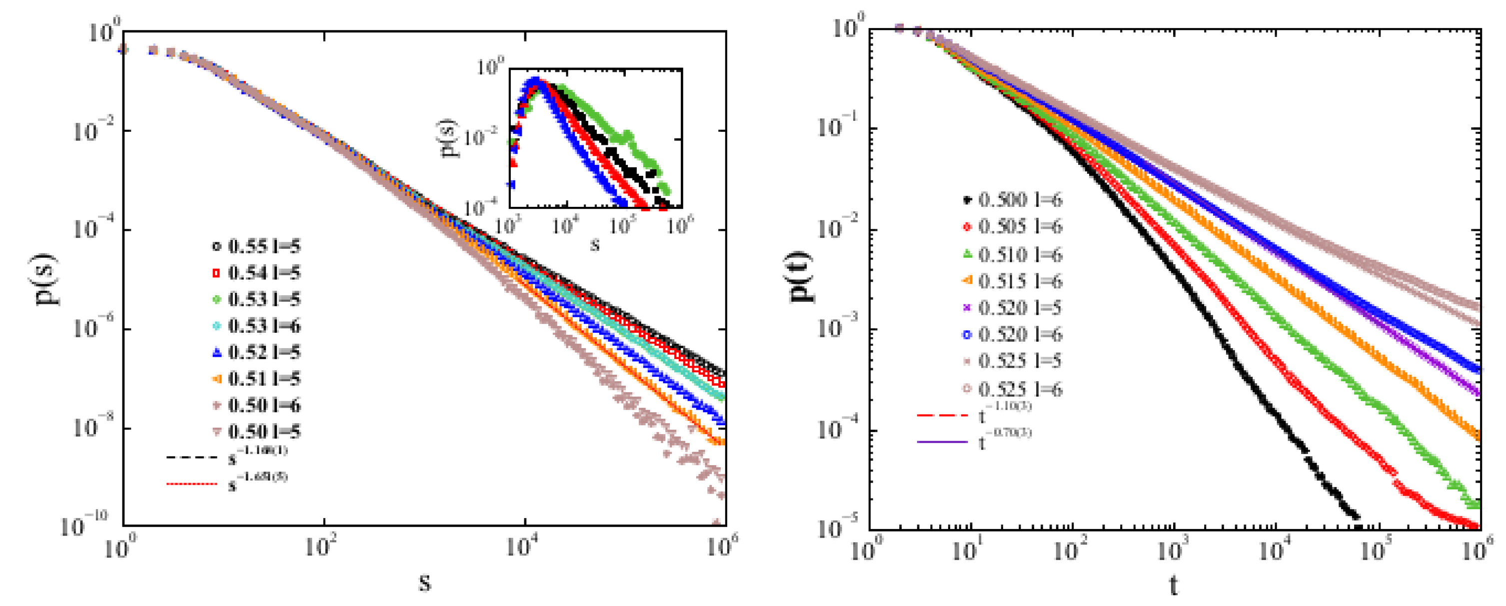

Fig. 3: Left: Avalanche size distributions. Inset: overlapping avalanches case for half-filled initial condition at $\lambda = 0.51, 0.515, 0.52, 0.525$ (bottom to top symbols). Right: survival probability of the activity. Dashed lines are PL fits for the tails of $\lambda = 0.505, 0.52$ curves.

Inhibitory-refractory Model: Nodes stay for a time $\Delta\,t$ in a refractory state following an activation; they cannot fall back immediately to deactivation.

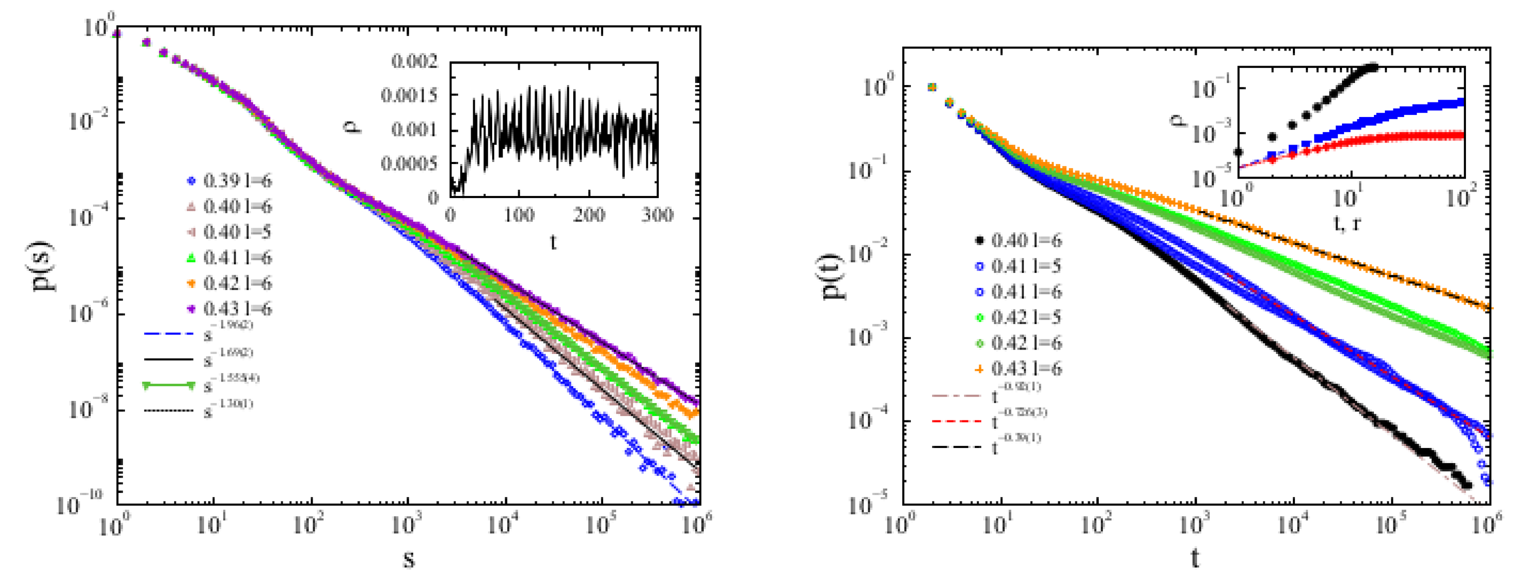

Fig. 4: PL fits for $t > 1000$. Left: The inset shows the oscillatory behavior of $\rho(t)$ of a single run for $\Delta\,t = 10$. Right: The inset shows $\rho(t)$ at $\lambda = 1, l = 7$ averaged over 10^5 realizations. Blue squares: excitatory; red diamonds: inhibitory. Black bullets: BFS $\rho(t)$ results. Dashed lines are PL fits for the initial regions: $1 \leq t < 10$) resulting in effective dimensions: $d_{eff} = 1.84(3)$ (excitatory), $d_{eff} = 1.19(1)$ (inhibitory), $d = 4.18(5)$ (with $\rho(t) \sim r^{d_{eff}}$ and graph dimension estimated for $5 < r < 10$).

- **Griffiths phase (GP) can occur in high dimensional systems due to fragmentation of the activity propagation caused by the modules**
- **Nonuniversal PLs suggest that Griffiths effects are present**

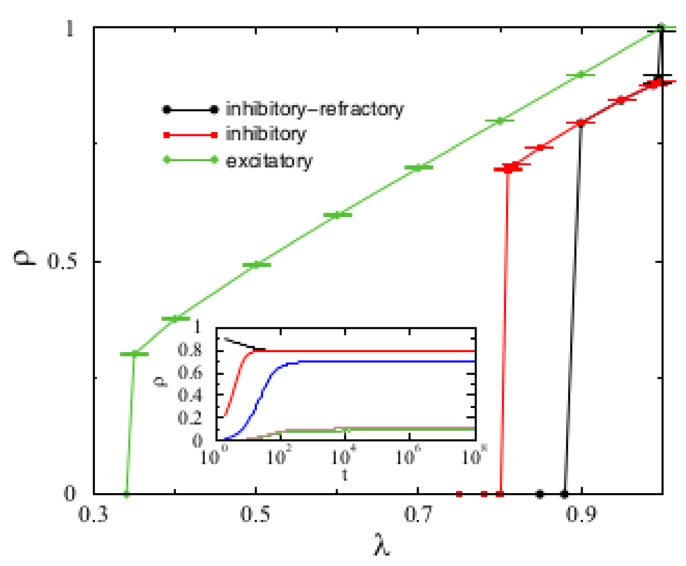

Fig. 5: Steady-state behavior for the excitatory, inhibitory, and refractory-inhibitory cases. Inset: evolution of ρ in an inhibitory HMN2d with $N = 4096$ for different initial activity densities: $\rho(0) = 0.0005, 0.001, 0.01, 0.1, 1$ (bottom to top curves).

- **Discontinuous jump in ρ, metastability and GP: Hybrid Phase Transition!**

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