Conformational Entropy Estimation of a Non-Markovian Random Walk by Autoregressive Generative Modeling

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Introduction

We propose a new flexible method of calculating conformational entropy of lattice polymer and random walk models and apply it to reconstruct phase diagram of a novel model of non-Markovian discrete random walk on a three-dimensional lattice, which we call volume and surface reinforced random walk.

Random Walk with Volume and Surface Reinforcement

Consider a random walk on some graph $\mathcal{G}$ defined by its sets of vertices and edges $\mathcal{G} = (\mathbb{V}, \mathbb{E})$. Let V_N be a set of vertices visited up to step N of the walk, and v_N – the position of the walker at step N. For each site v_i, which is neighbor of v_N, define a weight

$$w(v_i) = I(e_{N,i} \in \mathbb{E}) \times \begin{cases} \exp a, & \text{if } v_i \in V_N; \\ 1, & \text{if } v_i \notin V_N \text{ and } v_N \text{ is the only} \\ & \text{neighbor of } v_i \text{ from } V_N; \\ \exp b, & \text{if } v_i \notin V_N \text{ and there are} \\ & \text{other neighbors of } v_i \text{ in } V_N, \end{cases} \quad (1)$$

where $e_{N,i}$ denotes the edge connecting v_N and v_i, and I is the indicator function. Now the probability of a walker going from v_N to v_i on step $N+1$ is proportional to $w(v_i)$:

$$P(v_{N+1} = v_i) = \frac{w(v_i)}{\sum_k w(v_k)} \quad (2)$$

This model has two parameters: a (volume reinforcement) and b (surface reinforcement). Case of $b = 0$ was earlier discussed in [1, 2]. Our goal is to estimate conformational entropy of the model trajectories for $\mathcal{G} = \mathbb{Z}^3$.

Method of conformational entropy estimation

The method approximates the full-dimensional probability density function over all possible conformations by deep autoregressive generative model resembling the conditional PixelCNN [3]. The joint distribution $p(x|\theta)$ of a trajectory x conditioned on a vector θ of macroscopic parameters is expanded as

$$p(x|\theta) = \prod_{i=1}^{\text{len}(x)} p(x_i | x_{<i}, \theta), \quad (3)$$

where conditional distributions are modelled by a 3d fully convolutional neural network with locally masked weights.

We present an adaptive masking mechanism which takes into account sequential nature of the trajectory generation process and allows to define sparse three-dimensional generalization of conditional PixelCNN capable of computing all conditional distributions in a single forward pass. The network then is trained on the ensemble of trajectories generated for different macroscopic parameters using maximum likelihood estimation. Given a trained network we compute conformational entropy by averaging negative log-likelihood over ensemble of trajectories generated at fixed values of macroscopic parameters

$$S(\theta) = -\mathbb{E}_x \log p(x|\theta). \quad (4)$$

Adaptive masking mechanism

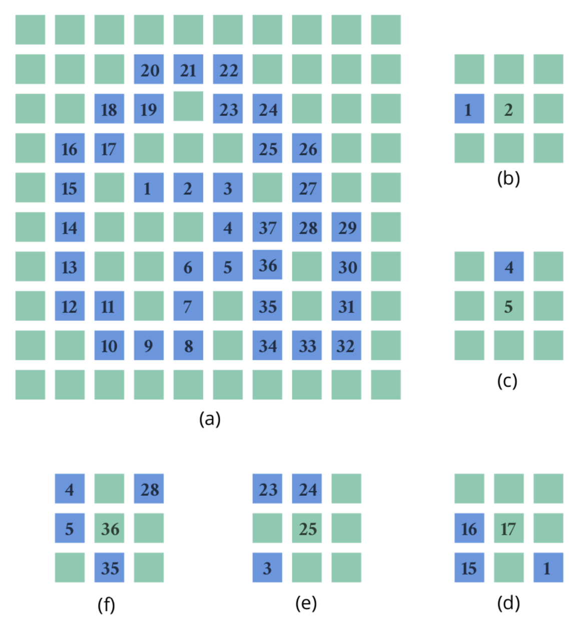

Figure: 1. Illustration of adaptive masking in 2d case. (a) Trajectory of a random walk, (b-f) Local trajectory-defined masks. Blue represents value of zero, and green represents one.

Adaptive masking generalizes the idea of locally masked convolutions proposed in [7] to the case of arbitrary orderings of voxels in 3d image. Since conditional probabilities in (4) should depend only on the previous points of trajectory, weights of 3d convolutional layers should be multiplied by a tensor with 0s for all trajectory points with the order larger or equal than the order of the current point. Points which are not part of the trajectory are also masked with zeros. Local masks in case of a 2d trajectory are illustrated by Fig. 1. Masks of this type are used only for the first convolutional layer. For second and further layers we remove the mask from the point itself and mask only the points of trajectory, which have the order strictly larger than that of the current point. For every point in a trajectory output layer returns positive real number representing probability of this point being added to the trajectory.

To scale up to larger trajectories instead of ordinary 3d convolutional layers we apply submanifold sparse convolutional layers [8] and store in memory only trajectory points and their neighbourhood.

Source code is avalible at github.

Characterization of Phases

- Phase (a) is a simple Brownian motion and has highest conformational entropy.
- In phase (b) the walk is mostly sticking to the surface of the already visited region, without either penetrating it or going away. As a result, visited volume grows, at least approximately, proportionally to the number of steps but the visited area forms a rather irregularly shaped blob with a very developed surface, which is reminiscent of the shapes of polymer rings in a melt [4, 5].
- Phase (d) is the phase of supercollapsed ball first predicted in [6] and well-studied in the volume-reinforced ($b = 0$) case [1, 2]. The visited region of the walk is in this case asymptotically a smooth ball with radius growing as $N^{1/4}$ with growing number of steps.
- Phase (c) is similar to phase d but for the fact that the supercollapsed ball in this case has a rough surface and is porous. As a result, the surface and the volume of the ball are growing as $N^{2\sigma}$ and $N^{3\nu}$, respectively, with $\sigma > \nu$.

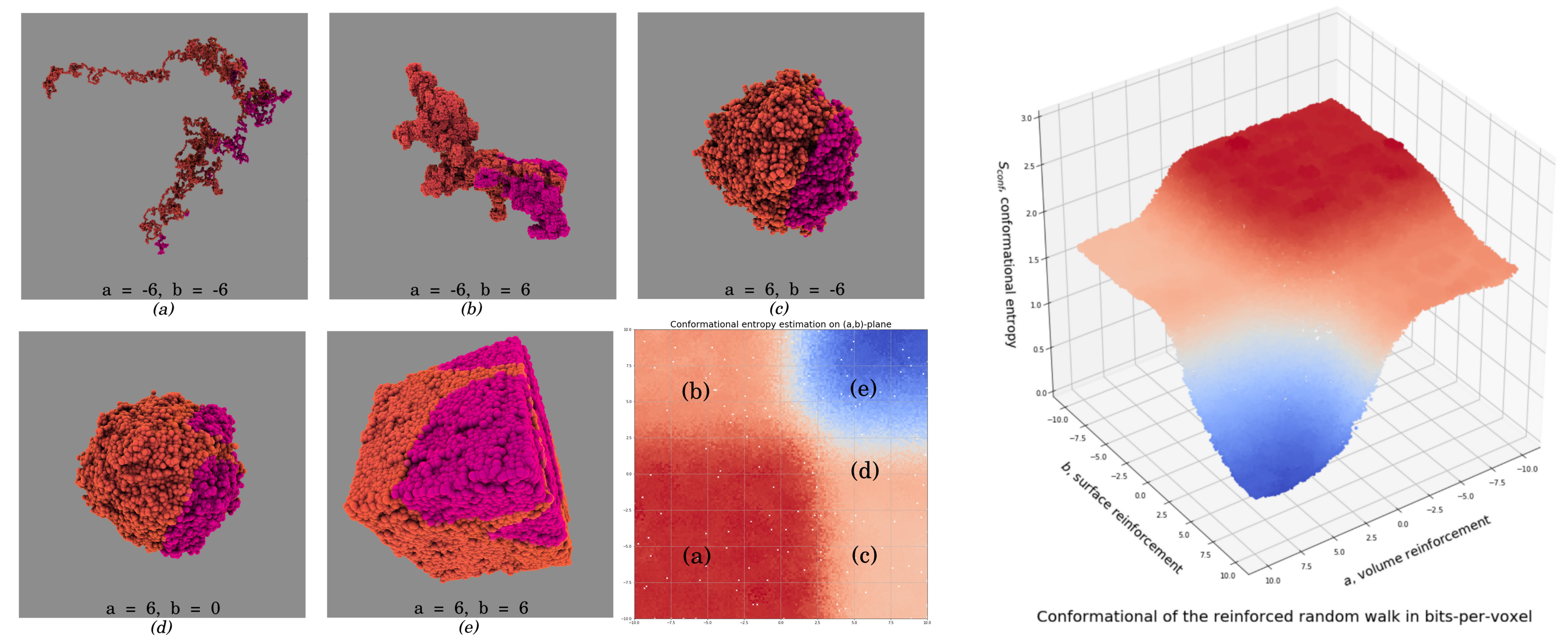

Figure: 2. Right: Examples of representative trajectories of the 5 phases of the model. Left: Plot of conformational entropy in bits-per-voxel estimated by 3d PixelCNN with adaptive masking.

- To understand phase (e) note that on the $a, b \gg 1$ limit it is beneficial for the visited volume to have extended flat faces, which grow without defects up to the area of order e^b. As a result, crystalline-like objects are formed. Clearly, for any given a, b they are unstable in the $N \to \infty$ limit, but the length of the walk needed for the defects to destroy this crystalline structure is exponentially large. However, the existence of this phase is a peculiarity of the simple cubic lattice, which is conducive to the flat face formation. We checked that phase diagram of a similar walk on body-centered cubic lattice does not include this phase. Instead, there is a direct transition from crumpled-globule-like phase (b) to supercondensed ball phase (d)

References

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