The Hybrid Potts (HP) model

- Consider a Potts-Hamiltonian \( H(p,q) \) where \( p \) is a configuration of \( N \) spins, each can select one out of \( \{1,2,\ldots,Q\} \) colors where \( Q_i = -X_i(q_i - q_0) + q_i \).

\[
\sum_{i=1}^{N} Q_i = \sum_{i=1}^{N} -X_i(q_i - q_0) + q_i = \cdots
\]

and \( X_i \) are i.i.d. random variables.

- Let \( q_i \) be the maximal integer for which the transition is continuous. Each spin can be colored by \( q_i \leq q \) colors (containing \( q_i - q_0 \) additional "weak" colors), with probability \( 1 - p \).

- Spins colored with probability \( p = 1 - q_0 \) belong to strong or weak regions, respectively.

The standard HP model

- Simple clusters, growing sub-exponentially with their size, have a minimal number of sites per bond. The expected number of such fractals is

\[
\frac{1}{q}(q_i) = \mu
\]

for \( q \leq p \).

- The large \( q \) behavior is then given by

\[
p = \frac{\ln q}{q}
\]

for \( q > p \).

The Hybrid HP model

- The free energy per site of simple clusters is given by

\[
\beta f = \frac{1}{\ln q} \ln q + (1 - p)\ln q + (1 + 2q) \times \left( \ln q - \ln q - n^* \ln n^* - (1 - n^*) \ln(1 - n^*) \right)
\]

\[
+ n^* \ln p^* + (1 - n^*) \ln(1 - p^*) + n^* \ln(q/q_0) = 0
\]

for \( p^* \).

- The large \( q \) behavior is then given by

\[
p^* = \frac{\ln q}{q}
\]

for \( q > p^* \).

The standard HP model

- Simple clusters, growing sub-exponentially with their size, have a minimal number of sites per bond, i.e., \( 1/2 + x/\omega \) \( (0 \leq x < 1) \) sites per bond. Let \( \psi(x, \omega) \) be the number of fractals with \( n \) sites such that \( k \) of them are positioned in strong regions. The expected number of such fractals is

\[
\langle n \rangle = \psi^*(n) = \psi(n) \left( \frac{\beta f}{2} \right)^{1-\frac{1}{\beta f}}
\]

where \( \sum_n \psi(n) \approx 1 \) and\( \mu = \psi^*(1) \) is the growth constant of those fractals [1].

- The expected change in the number of states is given by \( \psi(x, \omega) \psi^{-1} = \omega + x \). The free energy per site can be written as

\[
- \beta f_{\text{free}} = \beta f_{\text{free}} + \ln q - \frac{1}{q}(q_i) + \ln n_i
\]

- In the large \( n \) limit, the sum in \( (3) \) can be replaced with the maximum of the summand obtained at \( k \) satisfying

\[
\frac{\psi(x, \omega)}{\psi^*(1)} \leq \omega + x
\]

\[
(4)
\]

The mean-field (MF) model

- The MF Hamiltonian is given by

\[
H_{\text{MF}} = -\frac{1}{q}(q_i) - \frac{1}{\psi(x, \omega)} \psi(x, \omega)
\]

(10)

- Let \( \xi_i \) and \( \eta_i \) be the fraction of spins with color \( i \) \( \{1,2,\ldots,q\} \) in strong and weak regions, respectively. The number of states with energy \( E(\xi, \eta, q) \) is

\[
\Omega(\xi, \eta, q) = \frac{N!}{\left( \sum_{i=1}^{q} N_i! \right)}
\]

(11)

- Introducing the Lagrange multipliers \( \xi \) and \( \eta \), the free energy per site rate equals

\[
\frac{\beta f}{\lambda} = \sum_{i=1}^{q} \left( \xi_i + (1 - \psi(x, \omega)) \psi_i - \eta_i \right) + \frac{\lambda}{q} \sum_{i=1}^{q} \left( \xi_i \right) - 1 + \lambda \left( \eta_i - \eta \right)
\]

(12)

- The quantities \( \xi_i, \eta_i \) can take the form

\[
\xi_i = \frac{1}{q}(q_i - q_0) \quad \eta_i = \frac{1}{q}(q_i - q_0)
\]

(13)

Main result

There is a marginal concentration \( p^* \) such that the transition is discontiguous for \( p < p^* \) and continuous for \( p > p^* \).

References