Approach to consensus in the granular voter model

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Abstract

A model for continuous-opinion dynamics is proposed and studied by taking advantage of its similarities with a mono-dimensional granular gas. Agents interact as in the Deffuant model, with a parameter α controlling the persuasibility of the individuals. The interaction coincides with the collision rule of two grains moving on a line, provided opinions and velocities are identified, with α being the so-called coefficient of normal restitution. Starting from the master equation of the probability density of all opinions, general conditions are given for the system to reach consensus. The case when the interaction frequency is proportional to the β-power of the relative opinions is studied in more detail. It is shown that the mean-field approximation to the master equation leads to the Boltzmann kinetic equation for the opinion distribution. In this case, the system always approaches consensus, which can be seen as the approach to zero of the opinion temperature, a measure of the width of the opinion distribution. Moreover, the long-time behaviour of the system is characterized by a scaling solution to the Boltzmann equation in which all time dependence occurs through the temperature. The case $\beta = 0$ is related to the Deffuant model and is analytically soluble. The scaling distribution is unimodal and independent of α. For $\beta > 0$ the distribution of opinions is unimodal below a critical value of $|\alpha|$, being multimodal with two maxima above it. This means that agents may approach consensus while being polarized. Near the critical points and for $|\alpha| \geq 0.4$, the distribution of opinions is well approximated by the sum of two Gaussian distributions. Monte Carlo simulations are in agreement with the theoretical results.

Model

System

- N agents on a single-connected and undirected network with adjacency matrix A_{ij}
- The state/opinion of agent i is $s_i \in (-\infty, \infty)$
- The state of the system at a given time is $S \equiv \{s_i\}_{i=1}^N$

Dynamics

- Choose to agents i and j with a rate

$$\pi(s_i, s_j) = rA_{ij}\rho(s_i - s_j)$$

  where $r > 0$ is a constant and $\rho(x)$ is a non-negative function.
- The selected agents i and j with opinions s_i and s_j interact and change their opinions as

$$\begin{aligned} s_i &\to b_{ij}s_i \equiv s_i' \equiv s_i + \mu(s_j - s_i) \\ s_j &\to b_{ij}s_j \equiv s_j' \equiv s_j - \mu(s_j - s_i) \end{aligned}$$

  where $\mu \in [0, 1]$ is a parameter.

Properties

(i) Conservation of the "total opinion":

$$s_i' + s_j' = s_i + s_j$$

(ii) One consensus state:

$$s_i = \frac{1}{N}\sum_{k=1}^{N} s_k$$

(iii) Dissipation of "opinion energy":

$$(s_i'^2 + s_j'^2) - (s_i^2 + s_j^2) = -2\mu(1-\mu)(s_i - s_j)^2 \leq 0$$

Analogy with a granular gas

1. One-dimensional granular gas with

$$\mu = \frac{1+\alpha}{2}; \qquad \alpha \in [-1, 1]$$

2. But "stochastic dynamics"
3. Properties:
   (i) Conservation of linear momentum
   (ii) Zero-temperature state
   (iii) Dissipation of energy

Master equation and mean field

Master equation

The master equation for $p(S, t)$ the probability density of state S at time t is

$$\partial_t p = \sum_{i>j} (|\alpha|^{-1} b_{ij}^{-1} - 1)\pi(s_i, s_j)p$$

where b_{ij}^{-1} is the inverse of b_{ij}

From the maser equation the probability density p_i for s_i at time t reads

$$\partial_t p_i(s_i, t) = \sum_{\{j|j\neq i\}} \int ds_j\, (|\alpha|^{-1} b_{ij}^{-1} - 1)\pi(s_i, s_j)p_{ij}(s_i, s_j, t)$$

Mean opinion

The mean opinion

$$\bar{s} \equiv \frac{1}{N}\sum_i \langle s_i \rangle \qquad \Rightarrow \qquad \frac{d}{dt}\bar{s} = 0$$

Opinion temperature

The opinion temperature

$$T \equiv \frac{1}{N}\sum_i \left\langle (s_i - \bar{s})^2 \right\rangle \qquad \Rightarrow \qquad \frac{d}{dt}T = -\zeta T$$

with ζ being the so-called cooling rate

$$\zeta \equiv \frac{\mu(1-\mu)}{NT} \sum_{\{i,j|i\neq j\}} \int ds_i ds_j\, (s_i - s_j)^2 \pi(s_i, s_j)p_{ij}(s_i, s_j, t) \geq 0$$

Observations:
- T is a decreasing function of time
- S is a consensus state $\Leftrightarrow T = 0$
- An absorbing state ($\frac{d}{dt}T = 0$) is not necessarily a consensus state ($T = 0$)

Mean field

Two approximations:

(a) Homogeneity (exact for fully-connected networks):

$$\begin{aligned} &A_{ij} = 1, \\ &p_i(s, t) \simeq p(s, t) \end{aligned}$$

(b) Mean-field approximation ("Molecular chaos"):

$$p_{ij}(s_i, s_j, t) \simeq p_i(s_i, t)p_j(s_j, t)$$

Boltzmann kinetic equation for $f(s, t) \equiv p(s, t)$:

$$\partial_t f(s_i, t) \simeq \int ds_j\, (|\alpha|^{-1} b_{ij}^{-1} - 1)\pi(s_i, s_j)f(s_i, t)f(s_j, t)$$

Results for $\pi(s_i, s_j) = rA_{ij}|s_i - s_j|^\beta$

Mean-field theroy

A scaling solution:

$$f(s, t) = Ns_0^{-1}\phi(c); \qquad s_0 \equiv \sqrt{2T}; \qquad c \equiv \frac{s}{s_0}$$

For $\beta = 0$:

$$\phi(c) = \frac{2\sqrt{2}}{\pi\left[1 + 2c^2\right]^2}$$

For $\beta > 0$, ϕ can be approximated by the sum of two Gaussian distributions

Comparison against numerical simulations

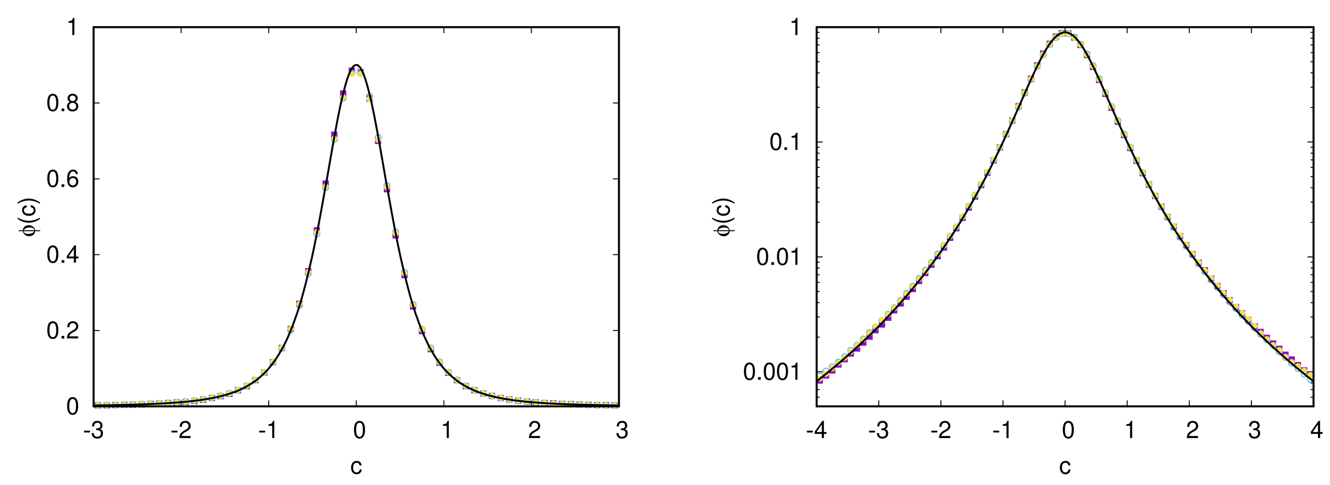

Figure 1: $\beta = 0$

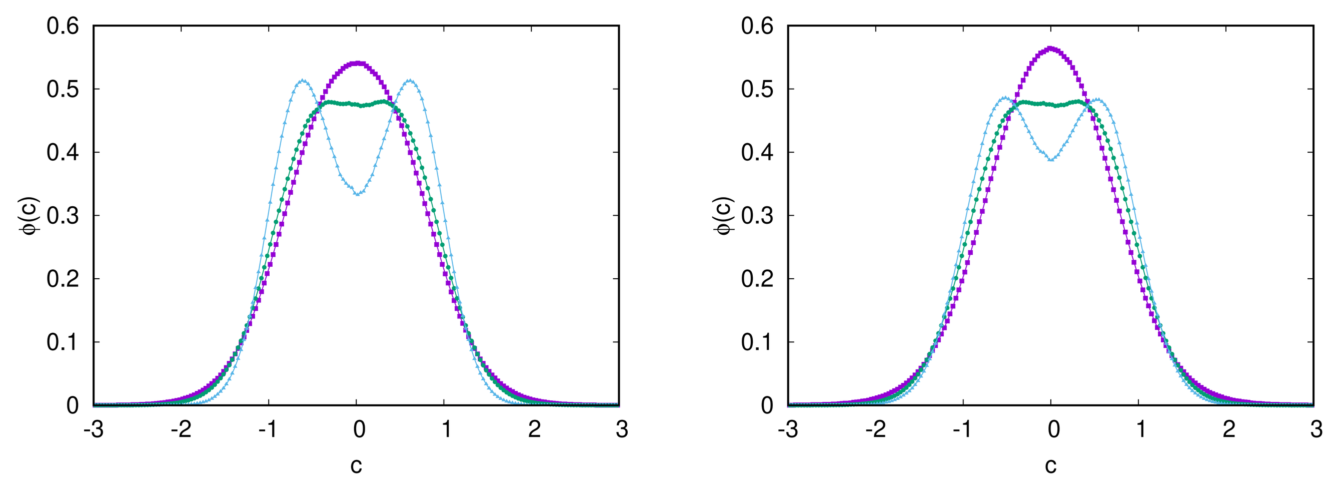

Figure 2: Left: $|\alpha| = 0.7$ (squares), 0.8 (circles), 0.9 (triangles), and $\beta = 1$. Right: $|\alpha| = 0.8$ and $\beta = 0.5$ (squares), 1 (circles), and 1.5 (triangles)

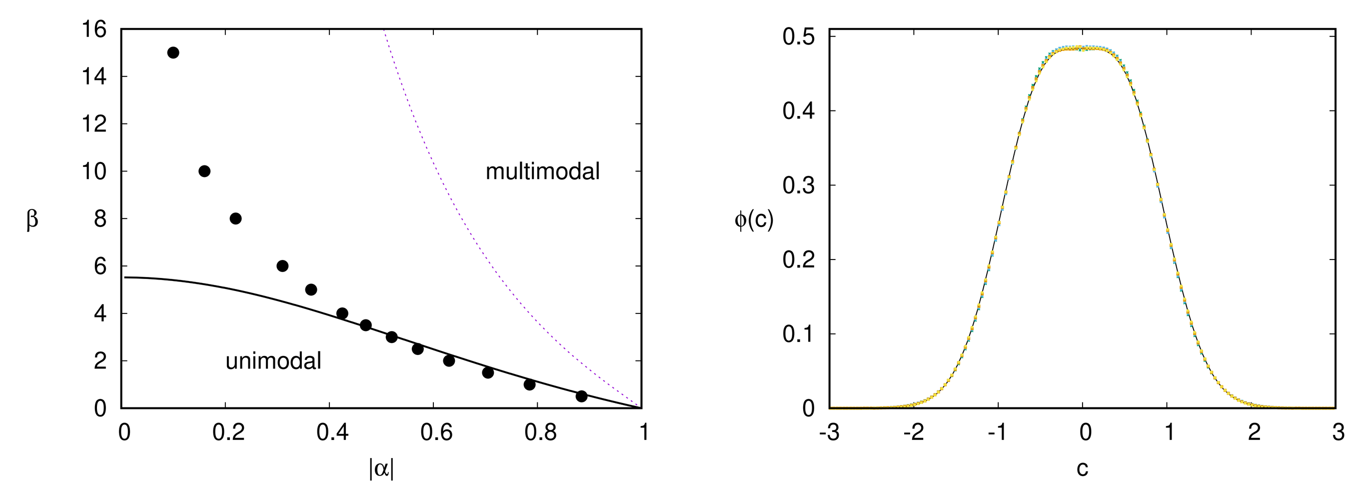

Figure 3: Left: phase diagram. Right: ϕ at the critical line

Conclusion

The individuals can split into two groups of opposite opinions as the system approaches consensus as a whole

References

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[2] Nagi Khalil. Approach to consensus in models of continuous-opinion dynamics: A study inspired by the physics of granular gases. *Physica A: Statistical Mechanics and its Applications*, 572:125902, 2021.