Development of a Low Variance Entropy Estimator and its Application to Quantify the Covid-19 Financial Markets Shock Response.

We improved **information entropy** estimation variance and consistency for (quasi-) continuous distributions with an adapted density discretization.

With it, we detect strong correlation of higher **inter-stock information transfer** with economic **policy uncertainty**, especially on the onset of COVID-19.

Introduction

- Important measures for non-linear data analysis such as transfer entropy or mutual information depend on information entropy.
- The Shannon Information Entropy is defined for discrete random variables:

$$H(X) = -\sum_{i=1}^{n} P(x_i) \log P(x_i)$$

- It can be thought of as an average surprise of the values $x_i \in X$.
- Generalisations for continuous data exist but suffer from problems. The entropy might be negative or contain diverging terms. Additionally, it depends on the often unknown underlying PDF.
- Hence, discrete information entropy is commonly used for entropy estimation on discretized continuous data.
- **Question**: How does discretization influence entropy value?

Methods

- Monte Carlo Analysis of samples of different length from 9 differently shaped distributions with added white noise.
- **Problem**: We cannot benchmark against a true $H(x)$ since it does not exist for continuous distributions. Thus, aim to minimize $\sigma(H(x))$.

Results

- We identify a strong dependence of the final entropy values of discretized continuous data on binning hyper-parameters.
- The adapted estimator improves variance and consistency for most binning methods.
- There is still residual dependence so the binning method has to be chosen carefully.

Adapted Estimator

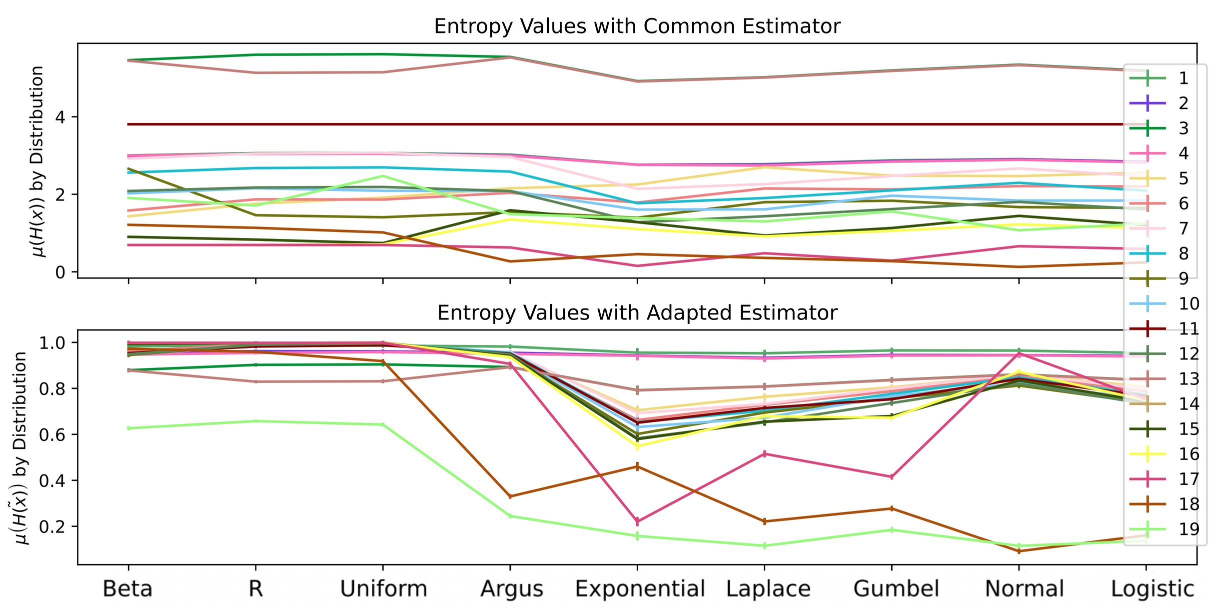
- **Motivation**: Probability mass should depend on bin data counts and bin size.
- With v_j, v the bin volume and total volume, n_j, n the bin and total count, m the number of bins and D the dimension:

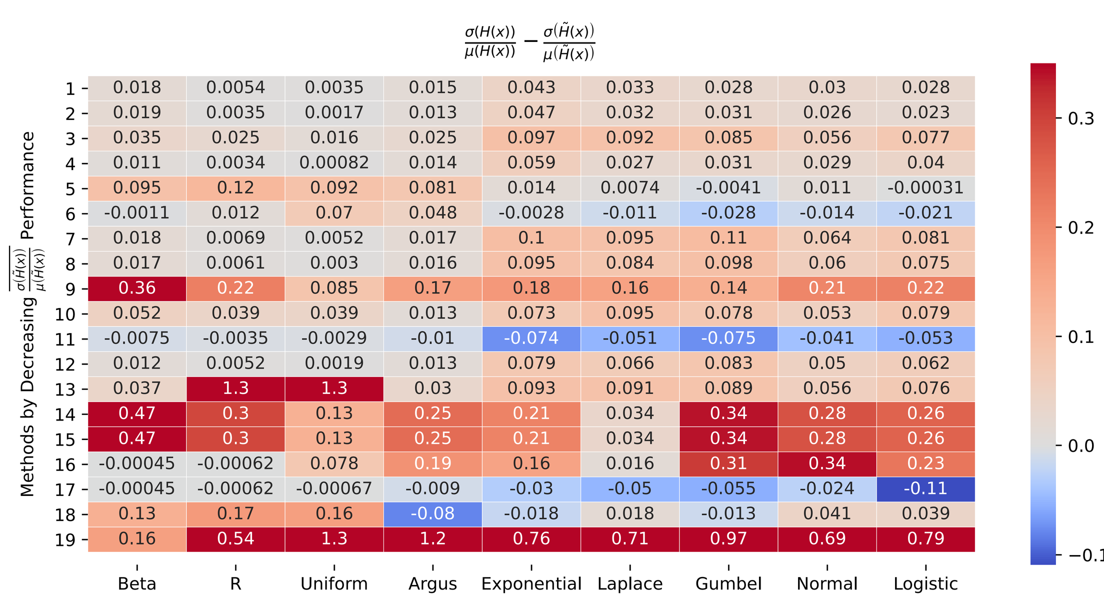
$$\hat{f}_n(x) = \frac{1}{C} \sum_{j=1}^{m} \frac{n_j}{n} \sqrt[D]{\frac{v_j}{v}} I_{x \in B_j}$$

$$\rightarrow p_j = \frac{1}{C} \frac{n_j}{n} \sqrt[D]{\frac{v_j}{v}}$$

- Normalise the entropy by number of bins:

$$\tilde{H}(x) = -\frac{1}{\log m} \sum_{j=1}^{m} p_j \log(p_j)$$





- More consistent results between binning methods for the adapted estimator
- Smaller variance of most binning methods with the adapted estimator

Application in Stock Network

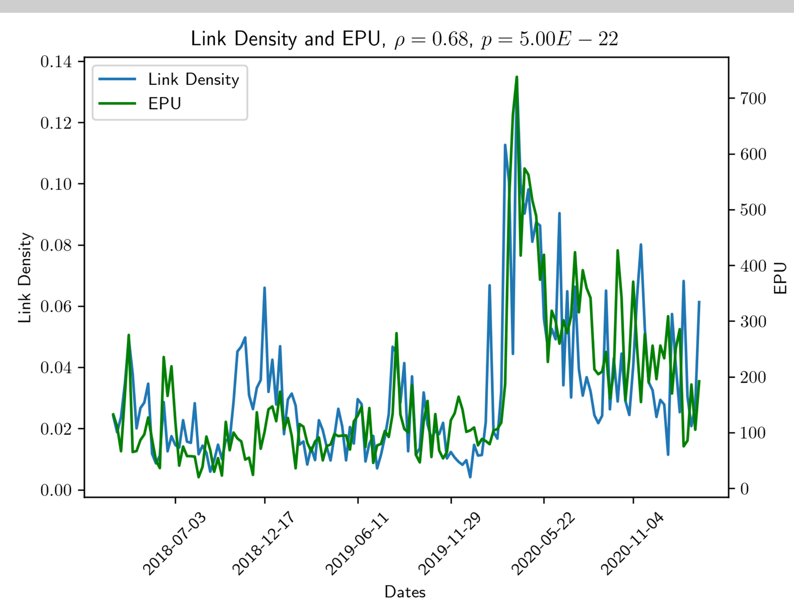
- Daily adjusted close prices of the 100 S&P500 constituents with the highest market capitalisation starting 2018.
- Calculate directed network with transfer entropy as edges utilizing our adapted entropy estimator.

$$TE_{x \to y} = H(Y_t, Y_{t-1}) + H(X_{t-1}, Y_{t-1}) - H(Y_t, Y_{t-1}, X_{t-1}) - H(Y_{t-1})$$

- Obtain time evolution by slicing into 50 day windows.
- Calculate link density of slices:

$$L = \frac{\sum_{i \neq j} TE_{i \to j}}{n^2}$$

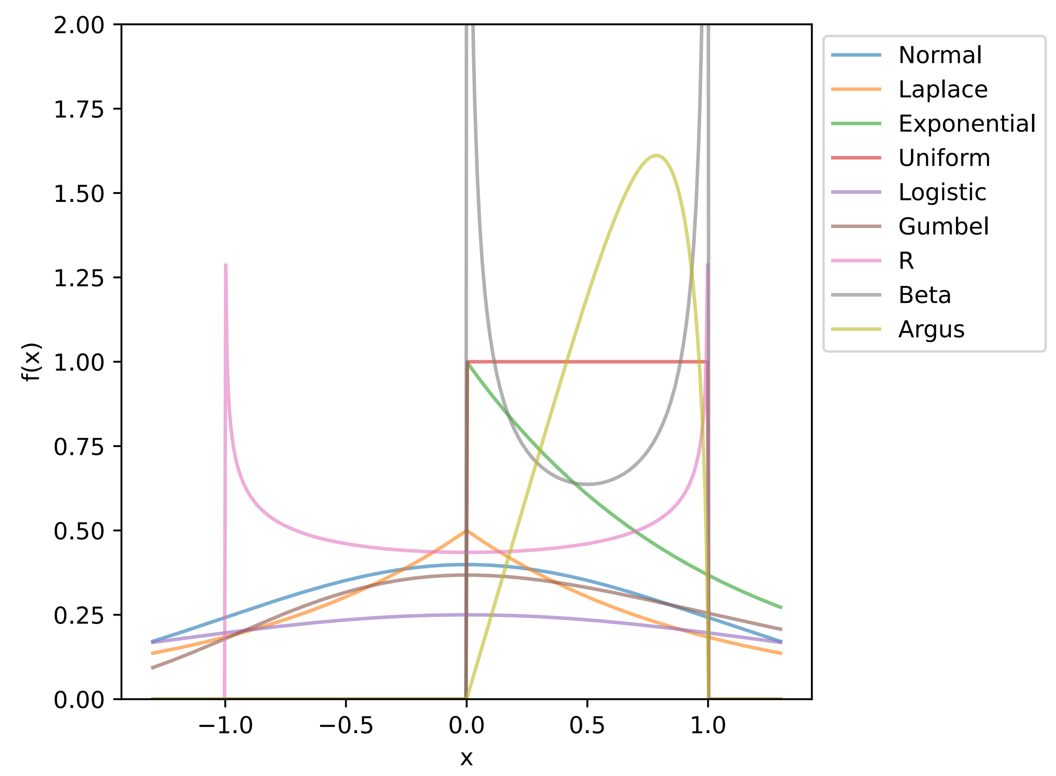
- Use the Economic Policy Uncertainty Index (EPU) by Baker et al. to quantify uncertainty.
- EPU indexes economic uncertainty by tracking the number of related articles in newspapers and the heterogeneity of opinions therein.



- We find strong correlation between inter-stock information transfer and EPU. Both EPU and information transfer are especially high during the COVID-19 onset in march 2020.

Binning Techniques

1. K-Means
2. Agglomerative Hierarchical Clustering
3. Minimizing Cross Validation
4. Expectation Maximisation with Gaussian Mixture Model
5. Freedman-Diaconis Rule
6. Scotts Rule
7. Square Root N Bins
8. Rice Rule
9. Knuths Method
10. Doanes Bins
11. Agostinos Uniform Probability
12. Sturges Bins
13. Shimazakis Bins
14. Akaike Information Criterion
15. Small Sample Akaike Information Criterion
16. Bayesian Information Criterion
17. Maximising Cross-Validation Likelihood
18. Mean Shift
19. DBSCAN



- We tested binning methods and the estimators against differently shaped distributions.

Properties of Estimator

- L an upper bound to the PDF and derivative of the PDF, M the size of the biggest bin, D the dimension, v the total volume of non-empty bins, C a normalisation constant and x^* a value in the bin range:

$$\mathbf{var}\left(\hat{f}_n(x)\right) \le \left(\frac{M}{v}\right)^{\frac{2}{D}} \frac{LM}{Cn}.$$

$$\mathbf{bias}\left(\hat{f}_n(x)\right) \le LM + L\left(\sqrt[D]{\frac{M^{D+1}}{C^D v}} - 1\right)$$

$$\mathbb{E}\left(\hat{f}_n(\vec{x})\right) = \frac{\sqrt[D]{v_j^{D+1}}}{Cv} f(\vec{x}^*)$$

Discussion

- Binning has a strong influence on dependent measures.
- Variance reduction with adaptation of measures is possible.
- However, the binning still residual influence on the final entropy value and should be chosen consciously and carefully.
- **Application**: Information transfer between stocks correlates highly with economic policy uncertainty.

References

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